Adv. Math. Fin. App., 2024, **9**(1), P. 321-335

Advances in Mathematical Finance & Applications
www.amfa.iau-arak.ac.ir
Print ISSN: 2538-5569
Online ISSN: 2645-4610
Doi: 10.22034/AMFA.2023.1967167.1797

Research Paper

Portfolio Optimization under Varying Market Risk Conditions: Copula Dependence and Marginal Value Approaches

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ARTICLE INFO
Article history:
Received 2022-09-06
Accepted 2023-02-09

Keywords:
Asset Portfolio
Extreme Value Theory
Copula, Market Risk

ABSTRACT
This paper aims to investigate the portfolio optimization under various market risk conditions using copula dependence and extreme value approaches. According to the modern portfolio theory, diversifying investments in assets that are less correlated with one another allows investors to assume less risk. In many models, asset returns are assumed to follow a normal distribution. Consequently, the linear correlation coefficient explains the dependence between financial assets, and the Markowitz mean-variance optimization model is used to calculate efficient asset portfolios. In this regard, monthly data-driven information on the top 30 companies from 2011 to 2021 was the subject to consideration. In addition, extreme value theory was utilized to model the asset return distribution. Using Gumbel's copula model, the dependence structure of returns has been analysed. Distribution tails were modelled utilizing extreme value theory. If the weights of the investment portfolio are allocated according to Gumbel's copula model, a risk of 2.8% should be considered to obtain a return of 3.2%, according to the obtained results.

1 Introduction

The traditional Markowitz portfolio theory assumes that the distribution of asset returns follows a multivariate normal distribution. However, the distribution of financial asset returns is frequently asymmetric and heavy-tailed [10]. Additionally, it has been observed that financial assets exhibit nonlinear characteristics. Therefore, in risk management, it is necessary to consider the tail risk of by the asset portfolio [15]. Left-tailed risk demonstrates the likelihood of unanticipated negative outcomes. When these adverse events occur, they have severe repercussions (extremely negative returns) that send price shock waves throughout the financial markets or to specific asset classes. Three standard deviations to the left of the mean, on the left side of the yield curve is the location of these outcomes [14].
After the emergence of financial crises and the subsequent severe negative returns, a new abnormality in the context of risk and expected return arose that could not be explained by the previously established anomalies [27]. Large negative returns due to unfavourable occurrences, such as financial crises, are positioned at the left end of the distribution function (close to three standard deviations from the mean).

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In the left tail, the relationship between risk and projected excess return is no longer positive. The extreme value is one of the new approaches for estimating the risk tail on the virtual currency market. Although the traditional parametric and non-parametric methods work well in the field of experimental distributions with many observations, they are inappropriate for the extreme tails of distribution and have disadvantages. This is due to the fact that extreme risk management requires estimating quantiles and tail probabilities, which are typically not directly observable in the data. Traditional risk management models cannot describe tail events because 1) they focus on the entire distribution and employ specific distributions, 2) they do not include the heavy tail aspect of the probability distribution, and 3) only a small portion of the data is in the tails [28]. According to the modern portfolio theory, diversifying investments in assets that are less linked with one another allows investors to assume less risk. In numerous models, asset returns are assumed to follow a normal distribution. Consequently, the linear correlation coefficient explains the interdependence of financial assets, and the mean-variance optimization model of Markowitz [29] is used to calculate efficient asset portfolios. Nonetheless, Embrechts et al. [12] and Forbes & Rigobon [30] discovered that the linear correlation coefficient is not a reliable indicator of the reliance between assets. In fact, a nonlinear relationship between financial assets is possible. To solve this deficiency and represent the dependence structure of multivariate data without making any special assumptions about the marginal distribution, the Copula theory was developed that has the advantage of separating the marginal distributions from the joint distribution's dependence structure. The asymmetry and dependency in the extreme tails of distribution can also be explained by considering the Archimedean copula pattern. Several scholars have utilized the Archimedean copula model in order to model the dependency structure between financial time series returns.
In the financial literature concerning the subject of stock portfolio optimization, the concept of dependence between the returns of different stocks is a crucial factor that influences the optimization. The majority of earlier studies focused the joint distribution of return. In the literature on financial economics, a different method that takes into account drawbacks like the linear correlation coefficient, asymmetry, and tail dependence in the distribution of financial returns has been proposed for modelling the structure of dependence between multivariate data without making any assumptions about marginal distributions. In the context of estimating the value at risk of an asset portfolio, the copula parameter has replaced the linear correlation coefficient in the financial literature over the past decade [6].
Accordingly,following Karma [19], the marginal value theory model is used in this research to optimize the asset portfolio, and various copula models are employed to model the dependence structure between asset return distributions. In addition, extreme value theory has been used to model the distribution of asset return. Copula is a versatile mathematical technique that combines a number of univariate marginal probability functions to yield a multivariate cumulative probability function. In fact, the copula is built on the nonlinear interaction and dependence between variables, and it connects the joint distribution and marginal functions. Copulas can be understood from two vantage points. On the one hand, they relate multivariate distribution functions to their one-dimensional marginal distribution functions, and on the other hand, they have uniform one-dimensional marginals throughout the interval [0-1]. Copulas can form virtually any type of marginal cumulative probability functions. Because, in order to construct a multivariate model, the marginal distributions can be chosen independently of one another, and it is not required that the marginal function adhere to a certain distribution, as is the case with two-variable distribution functions. In addition, copula can represent the variations in the degree of correlation between variables in different portions of the joint probability distribution, which cannot be observed with other methods of simulating random variables.

The innovation of this paper is in the optimal portfolio under different market conditions, including the ups and downs of the capital market. Also, in this study, the copula function was used to estimate the synchronicity dependence between sections. The remainder of this paper is organized as follows. Section 2 examines the theoretical research literature and past studies. In Section 3, the research methodology is provided, and Section 4 describes the research empirical model. Section 5 concludes the paper and gives some recommendations.

2 Literature Review

In order to determine the degree of dependence between indicators when analyzing the structure of dependence between different asset returns, it is crucial to select the proper criterion. In other words, the nature of dependence between financial returns, financial market conditions, and their impact on investment are crucial issues in applied financial economics, so understanding the relationships between financial assets is largely about how to invest in these assets. Identifying the nature of dependence between assets and financial markets is therefore one of the primary concerns of scholars. In the literature of financial economics, a method for modeling the structure of dependence between multivariate data without imposing assumptions on marginal distributions has been proposed; this method is based on copulas and incorporates the nonlinear correlation coefficient, asymmetry, and tail dependence among the distribution of financial returns in comments. In most research, copula models have been used to predict the value at risk of the asset portfolio, but in this study, the optimal asset portfolio will be calculated with this model. In this regard, the extreme distribution in tail risk and time dependency will be considered, and the optimal combination and optimal model for the asset portfolio, which have received less attention in earlier research, will be provided.

Determining the optimal portfolio based on the Markowitz model involves numerous complexities, such as a huge number of calculations and variables, so that in a market with N investment plans, a number of $\frac{(N^2)+3\mathrm{N}+2}{2}$ variables must be calculated. On the other hand, determining the effect of an investment's risk on the risk of the investment group necessitated the computation of covariance and correlation coefficients, which made the calculations tedious and time-consuming. Contrary to the model's assumptions, the return distribution of the plans does not necessarily follow the normal distribution, and in some circumstances, the standard deviation of the distribution cannot be computed (Islami Bidgoli, 1995). In other words, modern portfolio theory cannot always reflect the realities of investment markets based on risk and return. How to quantify such risks presents a challenge. By definition, extreme events are unusual, thus there are relatively few observations regarding them, and it is difficult to generate estimates based on these observations. In this approach, extreme risk estimates will be highly uncertain, and this uncertainty will be particularly apparent when searching for extreme risks not just within the range of observable data, but also much beyond it. For instance, a risk manager may be interested in assessing risks associated with extremely unusual events. One of these events is the collapse of financial markets owing to unforeseen shocks.

Attempts to tackle the problem of extreme values ultimately resulted in the formulation of the extreme value theory. The field of applied statistics known as extreme value theory was developed to solve such problems. This theory focuses on the distinction between extreme values and the theories that should be offered accordingly. It should not come as a surprise that extreme theory of values differs from the

well-known statistical ideas we have already studied. This is mostly due to the fact that statistical notions are frequently founded on the central limit theorem, to the extent that this subfield of statistics is known as central tendency statistics. The extreme values, however, are derived from the extreme value theorems. These theorems are used by extreme value theory to describe what distributions match the extreme data, as well as how to estimate the required parameters. The extreme value theory is significantly distinct from the conventional distributions related with central tendency statistics. In addition, its parameters are distinct and more difficult to estimate [1].
Utilizing extreme value theory while addressing extreme value offers benefits [26]:

- First, based on the preceding assertions, the distribution of time series data can only be estimated well towards the distribution's center because the majority of observations are concentrated in this region. On the other hand, extreme values are uncommon, and there are, by definition, few observations in the distribution's tails. Clearly, this makes it impossible to determine the behavior of tails using existing statistical distributions.
- Second, based on the studies, the presence of a heavy and especially aberrant tail in the distribution of financial returns is clear; hence, it seems more plausible to estimate statistical tails using non-parametric methodologies. In this circumstance, it is evident that putting a known statistical distribution on our observations is not appropriate. To estimate tails, extreme value theory is very useful in this situation.
- Third, there is always the potential that extreme fluctuations are generated by mechanisms that are structurally distinct from typical market behavior. An extreme observation may be prompted, for instance, by a significant default or a speculative bubble. The distributional features of the data may alter during these intervals. These structural modifications necessitate isolating the estimate of the tail from the remainder of the distribution. This is especially beneficial in situations where the residual of the density distribution is not required, such as in value-at-risk computations.

Christoffersen et al. [11] examined the influence of dependence structure based on copulas on the optimal stock portfolio. The findings of this study indicated that the employment of Archimedean functions and structural dependence in the returns of various currencies can have a substantial impact on the stock portfolio optimization. Krzemienowski & Szymczyk [20] investigated the optimization of the stock portfolio utilizing the value-at-risk strategy based on structural dependence based on Archimedean copulas. Their findings indicated that the aforementioned method of measuring the value at risk in comparison to the variance-covariance approach and other standard methods yields superior wealth accumulation results. Using the DCC-GARCH-Copula method, Han et al. [16] addressed the stock portfolio optimization. Their results of indicated that the optimization of copulas approach and dynamic conditional dependence using the GARCH method are superior to conventional ways for maximizing the value of a portfolio of stocks exposed to conditional risk. Karma [19] attempted to optimize various currency pairs through the AR-GARCH-EVT methodology and structural dependence. The research demonstrates that the adoption of extreme value theory of structural dependence improves the ratio of value at risk to return when optimizing a stock portfolio compared to the variance-covariance method.
Saham Khadem et al. [21] examined the optimization of the major stock index in various nations using the conditional value at risk method and the extreme value theory approach with conditional heterogeneity variance and structural dependence (GARCH-EVT-Copula). Employing the reweighting method and one-day forecasting, they came to the conclusion that it outperformed a number of other traditional models. Tehrani et al. [1] optimized the stock portfolio on the Tehran Stock Exchange using a meta-

heuristic method for shrimp categories based on several risk parameters. With the aid of the new algorithm of shrimp groups, an attempt was made in this study to solve the stock portfolio optimization problem and determine the efficient frontier. Also, the risk was analysed with three metrics of variance, semi-variance and predicted decline. This study presents the adjusted stock returns of the fifty most active stock companies from September 22, 2012 to September 22, 2017. In this paper, first the efficient frontiers of optimal portfolios were determined using the risk criteria of variance, semi-variance, and expected loss. The approximate resemblance of the three efficient frontiers demonstrated the algorithm's consistency in locating it. Then, the Sharpe ratios produced from the method of shrimp groups were compared to those acquired from the methods of colonial competition and particle accumulation, and it was found that the algorithm of shrimp groups was preferred. This study showed that the algorithm of shrimp groups outperformed other conventional algorithms in locating the efficient frontier and optimal portfolios, thus it may be replaced with these approaches to produce superior results.

Rezaei et al. [31] studied optimization using three-objective particle swarm algorithm. Initially, two non-dominated sorting genetic algorithm (NSGA2) and multi-objective particle swarm optimization (MOPSO) were employed to estimate the two-objective model of minimal variance and maximum efficiency in order to spot the optimal algorithm. Due to the superior performance of the MOPSO method, it was employed to estimate the three-objective model for maximizing stock returns, minimizing risk, and minimizing the number of stocks.

Lalegani & Zehtabian [9] studied the feasibility of optimizing the investment portfolio by minimizing the value exposed to conditional risk in the Tehran Stock Exchange using the copula model and simulated data. Numerous research on the world's financial markets demonstrate that the performance of the examined models can be greatly enhanced by using criteria pertinent to the data's structure and characteristics. In the meantime, the copula is one of the models that has received considerable attention for determining the relationships between the model's variables. In this study, the simulated data based on the correlation caused by the copula and the generalized Pareto distribution were evaluated on the optimization of the portfolio of industries index in the Tehran Stock Exchange with the goal of minimizing the value exposed to conditional risk. Based on the statistical test, this technique boosts the portfolio's performance greatly. Sina & Fallah Shams [4] examined the value-at-risk and copula-CVaR models for portfolio optimization on the Tehran Stock Exchange. In this study, an effort was made to provide a more efficient model for optimizing the investment portfolio, which would provide greater returns by taking investment uncertainty into account. To this end, the VaR model with the variance-covariance approach was compared with the Copula-CVaR model to spot the efficient frontier. The research period was from 2014 to 2018, and the statistical population consisted of the top 50 companies listed on the "Tehran Stock Exchange." To estimate Copula-CVaR, the error terms in time series of asset return distribution was first calculated and standardized using the ARIMA-GARCH model. Using the Copula t-Student, the marginal distributions of the assets were then computed. In the final stage, through Monte Carlo simulation, the yield of the assets was simulated and their CVaR value was obtained for a 10-day period; then, the optimal portfolio composition for different risk levels was found at the 95% and 99% confidence levels. The results suggested that the optimal stock portfolio was better constructed utilizing the combined model, i.e. the Copula-CVaR model.

Sabahi et al. [5] optimized the asset allocation of the investment portfolio. This study aimed to suggest ideal investment weights between the assets of the Tehran Stock Exchange, Bahar Azadi coin, the U.S. dollar, and Bitcoin by minimizing the value at conditional risk using the average-value at conditional risk approach. Due to the heavy tail of the yield distribution of financial assets, a technique beyond the threshold was employed to anticipate the tail distribution using extreme value theory. Copula's theory was also adopted, and the joint distribution of assets was examined regardless of data collinearity and normality. The structure of correlations between series as time-varying was modeled using the dynamic conditional correlation estimation. The risk exposure value was then calculated using the Mean-CVaR model, and the investment priorities and weights among assets on the Tehran Stock Exchange, gold coins, US dollars, and bitcoin were determined as optimal values. Using the daily data of the aforementioned asset index from October 2014 to April 2018, the investment efficient frontier was detected. The results indicated that at zero risk level (conditional risk-exposed value), the stock exchange received the most investment weight owing to minimal variance fluctuations, while at the highest risk level, cryptocurrency (Bitcoin) received the highest investment weight due to larger returns. Similarly, the comparison of optimal portfolios using the conditional Sharpe ratio suggested that varied portfolios perform better than single assets, and the portfolio containing coins with an allocation of more than 70% and equal weights of dollars and bitcoins had the best performance. Also, according to the conditional Sharpe ratio, the minimum weight of coins in the best portfolio was 60%, while the maximum weight of dollars and bitcoins was 20%.

Tahani et al [32] study the worst case GARCH-Copula CVaR approach for portfolio optimisation. This paper addresses portfolio optimisation complexities by applying the Worst Case GARCH-Copula Conditional Value at Risk (CVaR) approach. In particular, the GARCH-copula methodology is used to model the portfolio dependence structure, and the Worst Case CVaR (WCVaR) is considered as an alternative risk measure that is able to provide a more accurate evaluation of financial risk compared to traditional approaches. Copulas model the marginal of each asset separately (which may be any distribution) and also the interdependencies between assets This allows an accurate risk to investment assessment to be applied in order to compare it with traditional methods. In this paper, present two case studies to evaluate the performance of the WCVaR and compare it against the VaR measure. The first case study focuses on the time series of the closing prices of six major market indexes, while the second case study considers a large dataset of share prices of the Gulf Cooperation Council's (GCC) oil-based companies. Results show that the values of WCVaR are always higher than those of VaR, demonstrating that the WCVaR approach provides a more accurate assessment of financial risk.

Andrew et al [25] investigated the Impact of typically involves ranking and selecting assets based on a non-financial impact factor, such as the environmental, social, and governance (ESG) score and the prospect of developing a disease-curing drug. They develop a framework for constructing optimal impact portfolios and quantifying their financial performances. Under general bivariate distributions of the impact factor and residual returns from a multi-factor asset-pricing model, the construction and performance of optimal impact portfolios depend critically on the dependence structure (copula) between the two, which reduces to a correlation under normality assumptions. More generally, we explicitly derive the optimal portfolio weights under two widely-used copulas---the Gaussian copula and the Archimedean copula family, and find that the optimal weights depend on the tail characteristics of the copula. In addition, when the marginal distribution of residual returns is skewed or heavy-tailed, assets with the most extreme impact factors have lower weights than non-extreme assets due to their high risk. They framework requires the estimation of only a constant number of parameters as the number of assets grow, an advantage over traditional Markowitz portfolios. Overall, these results provide a recipe for constructing and quantifying the performance of optimal impact portfolios with arbitrary dependence structures and return distributions. Sahamkhadam et al [24] extend the Black-Litterman (BL) approach to incorporate tail dependency in portfolio optimization and estimate the posterior joint distribution of returns using vine copulas. Our novel copula-based BL (CBL) model leads to flexibility in modelling returns symmetric and asymmetric multivariate distribution from a range of copula families. Based on a sample of the Eurostoxx 50 constituents (also for S&P 100 as robustness check), we evaluate the performance of the suggested CBL approach and portfolio optimization technique using out-of-sample back-testing. The empirical analysis and robustness check indicate better performance for the CBL portfolios in terms of lower tail risk and higher risk-adjusted returns, compared to the benchmark strategies.

3 Research Methodology

The objective of this study is to optimize the stock portfolio based on structural dependence factors via copulas and extreme values. The population of interest in this study is the Tehran Stock Exchange. The analysed sample also contains 30 companies with the greatest presence among the top 50 companies from 2011 to 2021, with monthly data frequency. The extreme value theory is a highly effective framework for analysing the behaviour of distribution tails. The central limit theorem discusses the volatility of cumulative sums and the centrality of the data distribution. Unlike the central limit theorem, the extreme value theory (EVT) is concerned with the maximum fluctuations of the sample and is particularly useful for modelling the probability distribution of tails that extend beyond the most severe observed fluctuation. There are generally two ways to determine extreme data. The first method is the generalized extreme value, also known as the block maxima method.

The focus of extreme value theory is maximal values. Regardless of the original distribution of the observed data, Fisher and Tippett's theorem states that the asymptotic distribution of the maxima corresponds to one of the Frechet, Gumbel, or Weibull distributions. These three distributions can be represented in a single equation (Equation 1):

$$H_{\xi}(x) = \begin{cases} exp\left\{-(1+\xi\left(\frac{r-\mu}{\sigma}\right)^{-1/\xi}\right\} & , if\ \xi \neq 0 \ \&\ 1+\xi\left(\frac{r-\mu}{\sigma}\right) > 0 \\ exp\left\{-\exp\left(-\left(\frac{r-\mu}{\sigma}\right)\right)\right\} & , \quad if\ \xi = 0 \end{cases} \quad (1)$$

Where μ is the location parameter of the distribution (mean), , is the scale parameter (dispersion), and ξ is the shape parameter. A way for obtaining extreme values from a sample of observations $X_t, t = 1,2,\dots,n$ with a distribution function as $F(x) = \Pr\{X_t \leq x\}$ is to spot the values exceeding a certain high threshold like u. When $X_t > u$, the threshold u increases for each t in $t = 1,2,\dots,n$. The additional value of u is calculated through $Y = X_i - u$. This method is referred to as the peaks over threshold (POT) method. The probability distribution of extra values X of u (i.e. y) is defined as follows, given the threshold value of u:

$$F(y) = \Pr\{X - u \leq y | X > u\} \quad (2)$$

Which indicates the probability of exceeding the value of X from the maximum threshold of u to the value of y if X exceeds u. This conditional distribution can be stated as Equation 3:

$$F_u(y) = \frac{\Pr\{X-u\leq,\ \ X>u\}}{\Pr(X>u)} = \frac{F(y+u)-F(u)}{1-F(u)} \quad (3)$$

Since for $x > u$ and $x = y + u$, the following statement can be written:

$$F(x) = [1-F(u)]F_u(y) + F(u) \quad (4)$$

This statement holds only when $x > u$. If N_u represents the number of observations above the threshold u and T represents the total number of observations in the sample, the probability distribution of values below the threshold can be approximated as follows:

$$\hat{F}(u) = \frac{T - N_x}{T} \tag{5}$$

The probability distribution of x values can therefore be expressed as follows.

$$\hat{F}(x) = \left[1 - \frac{T - N_u}{T}\right] F_u(y) + \frac{T - N_u}{T} = 1 + \frac{N_u}{T}\left[F_u(y) - 1\right] \tag{6}$$

A theorem developed by Balkema-De Haan (1974) and Pickands (1975) demonstrates that for sufficiently large threshold values u, the excess distribution function can be approximated by the generalized Pareto distribution (GPD).

$$G_{\xi,\sigma,v}(x) = \begin{cases} 1 - \left[1 + \xi\left(\frac{x-u}{\sigma}\right)\right]^{-1/\xi} & , if\ \xi \neq 0 \\ 1 - exp\left[-\left(\frac{x-u}{\sigma}\right)\right]^{-(x-v)/\sigma} & ,\ if\ \xi = 0 \end{cases} \tag{7}$$

When ξ approaches 0, the limit of the first relationship in the above equation is identical to Equation 2; therefore, the GPD can only be represented by the following equation:

$$G_{\xi,\sigma,u}(x) = 1 - \left[1 + \xi\left(\frac{x-u}{\sigma}\right)\right]^{-1/\xi} \tag{8}$$

So, the associated probability density function can be expressed as follows, where σ is a positive scale parameter and ξ is the shape parameter for heavy-tail distributions in financial time series.

$$g_{\xi_i,\sigma_i}(y_i) = \frac{1}{\sigma_i}\left(1 + \xi_i \frac{y_i}{\sigma_i}\right)^{\frac{1}{\xi_i}-1},\quad y_i = x_j - u_j \tag{9}$$

Maximum likelihood is used to estimate the aforementioned parameters. After approximating the tail risk of financial assets, the copula is used to investigate the time dependence between the return and risk of these assets. Due to the difficulty of calculating multivariate copulas, the bivariate copula model is created and its findings are utilized for multivariate copulas. A bivariate copula $C(u_1.u_2)$ is defined as the cumulative distribution function for a bivariate vector with a range of [0.1], with a uniform marginal distribution. If $C(U_1.U_2)$ is a bivariate vector, the copula is defined as follows (Raghfar & Ajorlou, 2016).

$$C(u_1.u_2) = P(U_1 \leq u_1.U_2 \leq u_2) \tag{10}$$

Presuming familiarity with marginal distribution functions (Equation 11):

$$F_i(x_i) = P(X_i \leq x_i).\quad i = 1.2 \tag{11}$$

Then, utilizing the transformation $U_i = F_i(X_i)$, a new bivariate function is generated using the marginal function of x_1 and x_2 (Equation 12):

$$F_i(x_1.x_2) = C[F_1(x_1).F_2(x_2)] \tag{12}$$

Sklar [33] demonstrated the reverse, so that if we have a bivariate distribution function, such as F, we may use the copula to determine its marginal distribution. Moreover, if the marginal distribution of F is considered to be continuous, the uniqueness of the copula C can be demonstrated. In Equation 12, the definition of the copula is stated as a cumulative distribution function. Assuming that functions E and

C are differentiable, the joint density function $f(x_1.x_2)$ has the form of Equation 13, where f(x) is the density function corresponding to F and we have density function Equation 13 [34].

$$f(x_1.x_2) = f_1(x_1) \times f_2(x_2) \times c[F_1(x_1).F_2(x_2)] \quad (13)$$

$$c(u_1.u_2) = \frac{\partial C(u_1.u_2)}{\partial u_1 \partial u_2} \quad (14)$$

This study employs copulas from the Normal, T-Student, and Archimedean copula families, including Frank, Gumbel, and Clayton. Ling (1965) first described the Archimedean copula family. The most significant characteristic of Archimedean copulas is that they are not elliptic copulas and permit modelling of various sorts of dependency arrangements.

4 Data Analysis

4.1 Statistical Characteristics of Research Variables

According to the theoretical literature and the research aims, the following question was investigated:
Is there a time dependence between the risk and return of the asset portfolio stocks?
How much tail risk is expected based on the extreme value theory?
Descriptive statistics is the organizing and classification of data, as well as the graphical representation and computation of values such as mode, mean, median, etc., that indicate the characteristics of the society's members. In descriptive statistics, the information acquired from a group describes only that group and, thus, it cannot be extrapolated to other groups. In descriptive statistics, tables and graphs are typically used to summarize data. The descriptive statistics of the model variables are displayed in Table 1, including the mean, standard deviation, etc.

Table 1: Descriptive statistics of efficiency index of sample companies

Variable	Mean	Standard deviation	Skewness	Kurtosis	Jarque-Bera statistic	Probability
Sample efficiency	0.0032	0.0189	5.56	6.41	34.85	0.000
Market efficiency	0.0010	0.0131	0.72	4.89	28.92	0.000

Source: Research findings

The statistical characteristics of the efficiency of the sample, which consists of 30 enterprises, and the efficiency of the market are presented in Table 1. The statistical distribution of the variables was explored by calculating the mean of the observations using the Jarque-Bera test statistic and the specified significance level for this statistic. Based on the claimed significance level, the acquired results revealed that the distribution of the employed variables is not normal.

4.2 Estimation of Extreme Value

Following is a discussion of the assessment of risk and return based on non-parametric and extreme techniques. In order to accomplish this, the extreme value index is computed using the following period-related information:

$$ES = -P_{t-1}(h\mu_t - \sqrt{h}\delta_t \frac{\varphi f^{-1}(x)}{x}) \quad (15)$$

Where $f^{-1}(x) = Z_\alpha$ represents the standard normal distribution of observations. If $f^{-1}(x) = t_{\alpha,v}\sqrt{\frac{v-2}{v}}$ has a t-Student distribution, then its degree of freedom corresponds to the condition in which the degree of freedom is conditional on $\frac{3(v-2)}{v-4} \le 5$.

$$I = \begin{cases} 1 & if \ |u| \le 1 \\ 0 & otherwise \end{cases} \quad (16)$$

Risk and return are computed with extreme value approach based on the above equation, which is determined by using the variance and stock returns to determine the extreme value.

Table 2: The Results of Variance and Efficiency Estimation to Calculate EVT

Variables	Coefficient	Significance level
Y-intercept	0.283	0.226
First lag of return	0.523	0.000
Second lag of return	-0.134	0.000
Gamma distribution	0.489	0.000
Tail of fluctuations parameter	-0.325	0.000
Fluctuations lag	0.452	0.000
Goodness-of-fit statistic	Coefficient of determination: Durbin Watson statistic:	0.68 2.21

Source: Research findings

As can be observed, the sum of the variance section coefficients is less than one, indicating that the calculated model is stable. After estimating the parameters of the generalized distribution, it is straightforward to calculate the average waiting time and minimum yield below/above a certain threshold. As an example, the average waiting time together with the distribution shape parameter, the μ location parameter, and $\sigma_{\max}$ the scale parameter have been precalculated and displayed in the table below.

Table 3: Results of Calculating the Average Waiting Time and Minimal Efficiency Yields

X	ξ	$P(r<a)$	$E[l(u)] = P(r>a) = 1-P(r<a)$	$P_j = 1/P(r>a)$
-0.05	0.212	0.999	0.001	1000
-0.04	0.212	0.998	0.002	500
-0.03	0.212	0.989	0.011	90.90
-0.02	0.212	0.985	0.015	66.66
-0.015	0.212	0.968	0.032	31.25
-0.01	0.212	0.948	0.052	19.23
-0.005	0.212	0.875	0.125	8
-0.001	0.212	0.067	0.933	1.07
0	0.212	0.003	0.997	1.003
0.001	0.212	0.019	0.981	1.07
0.005	0.212	0.429	0.571	8
0.01	0.212	0.593	0.407	19.23
0.015	0.212	0.778	0.212	31.25
0.02	0.212	0.897	0.103	66.66
0.03	0.212	0.917	0.083	90.90
0.04	0.212	0.983	0.017	500
0.05	0.212	0.994	0.006	1000

Source: Research findings

According to the stated contents, $E\left[l\left(u\right)\right]$ specifies the number of days until the stock price above the threshold U . After estimating $E\left[l\left(u\right)\right]$, it is possible to spot the minimum yield below a certain threshold.The minimum yield below a given threshold can be determined after estimation. In the above table, P_j represents the probability that the daily return index will surpass the minimal barrier before j. j is assigned equal to 1 since we are computing the marginal return for a defined time period of one day.

4.3 Copula-GARCH Function Estimation

To estimate the return and risk of the asset portfolio, dynamic conditional correlation (DCC) models, one of the forms of multivariate GARCH, are utilized in the following. Multivariate GARCH models offer the benefit of including the time-varying systemic risk exposure of each financial corporate. After the marginal distribution of F_i for the data has been obtained, the copulas are estimated. Gumbel's copula with GARCH-DCC marginal distribution with the lowest Akaike and Schwartz information requirements and the highest value for the likelihood function has the best performance among other functions for interpreting the dependence between two time series. The following are the outcomes of Gumbel's copula estimations:

Table 3: Results of Estimating the Parameters of the Copulas

Variables / **Sample**	**Model coefficients**			
	Gumbel's copula dependence coefficient (θ)	**Lower dependence coefficient of the distribution (λ_U)**	**Upper dependence coefficient of the distribution (λ_L)**	**Maximum likelihood function**
30 companies	1.73	0	0.69	77.58
Total index	1.36	0	0.29	64.69

Source: Research findings

The results reveal that the efficiency of the sample companies and the total index depend more on the distribution's upper tail. Consequently, these indicators are more dependent on good returns than they are on negative returns. After obtaining the residuals using the conditional standard deviation, they are then normalized. These residuals have a mean of zero and a variance of one. The cumulative distribution function for each company has been constructed using standard residuals, the Gaussian kernel method for the inner and smooth portion of the distribution function, and the extreme value method to estimate the upper and lower tails. The graph associated with the sample under examination is depicted in the figure below.

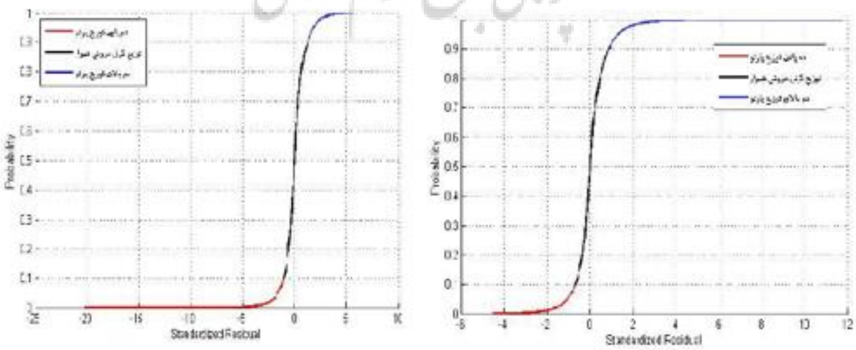

Fig. 1: Cumulative Distribution Chart of 30 Companies and Total Market Index

Source: Research Findings

The interpretation of the accompanying table and graph is that a risk of 8.2% must be accounted for in order to obtain a return of 2.3% if the weights of the investment portfolio are assigned as seen based on the estimated Gumbel's copula model. Considering the computation of tail risk, the estimation and calculation of the portfolio's efficient frontier are discussed below.

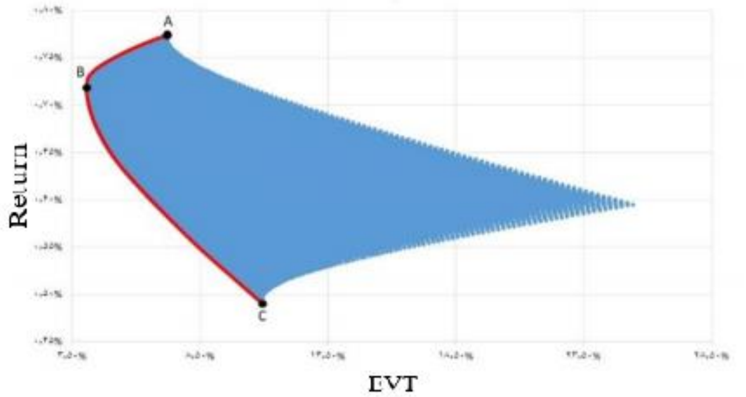

Fig. 2: Efficient Frontier of the Portfolio Using the Estimation of Tail Risk and Return
Source: Research Findings

According to Figure 2, the distance from A to B indicates the efficient frontier, but the distance from B to C indicates the inefficient frontier. Point B indicates the lowest risk.

5 Conclusion and Recommendations

The risk-return characteristics of asset allocation models are crucial and significant for investors. Traditional portfolio theory evaluates portfolio risk using linear correlation coefficients and standard deviation under multivariate normal distribution conditions. In order to get the optimal portfolio, the portfolio risk is minimized based on return levels. Pearson's correlation coefficient is typically employed as the basis for calculating correlation coefficients in the application of optimization in practice and experience. This criterion measures the linear link between normally distributed variables. Specifically, copulas can also explain the dependence structure of distributions that are not normal. In addition, these functions are quite versatile and can be used to analyse linear, nonlinear, and tail relationships. In this work, the expected loss model was employed to optimize the asset portfolio, while the Gumbel copula model was employed to model the dependence structure between asset return distributions. In this regard, information on the top 30 corporations from 2011 to 2021 was utilized. In addition, extreme value theory was utilized to model the asset return distribution.

Utilizing Gumbel's copula model, the dependence structure of yields was examined. Distribution tails were modelled with extreme value theory. The results indicated that if the weights of the investment portfolio are assigned according to Gumbel's copula model, then the risk should be 2.8% in order to obtain a return of 2.3%. On the basis of the obtained results, it is recommended that, since investors typically focus on the negative aspect of risk and view negative fluctuations as undesirable, these methods can be utilized in the majority of financial fields to determine optimal portfolios and predict the maximum loss of various assets and the asset portfolio consisting of these assets. Consequently, the EVT-GARCH-Copula technique can be utilized as an effective, efficient, and dependable instrument for optimizing other types of assets. Initial consideration should be given to equities with a greater

average yield by capital market investors. In addition, the amount of confidence considered in computing the value at risk has no effect on the optimal stock weights, so investors can consider a level of confidence in the computation of this risk measure and confidently optimize their asset portfolio.
The main goal in portfolio management is to help the investor in choosing the optimal portfolio according to his preferences and interests as well as the decision environment. Due to two major weaknesses in the Markowitz model, which includes the calculation and the problem of not considering the investor's interests, Sharp presented a single-index model in 1963, and his model also failed because he saw portfolio risk in only one factor. Favorable for investors. In this regard, the APT model solved the problem of investment risk to some extent, but due to not specifying the number of influencing factors, it was not well received in the field of practice. Our suggestion for future research is that researchers use the ideal planning model, because it can overcome the stated problems.

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