Properties of Optimal Consumption under Liquidity Constraints: New Results by Control Theoretic Approach

Masoud Derakhshan[*]

Optimality conditions for consumption behavior with liquidity constraints are obtained using the functional recurrence equation in Bellman's dynamic programming and the generalized Hamiltonian function in Pontryagin's maximum principle. The rejection of Hall's random walk hypothesis is then established for liquidity constrained consumers. An explicit mathematical relation is formulated which demonstrates the effects of liquidity constraints on consumption, which implies that under certain conditions the liquidity constraint may shift the optimal consumption profile forward even when the rate of time preference exceeds the interest rate. Our analysis is further developed to time-varying interest rates. Using the Kuhn-Tucker conditions, we have shown the interactions between the time-varying interest rate, the utility discount rate and the severity of liquidity constraints. It is shown, using the coefficient of absolute risk aversion, that how the time-varying interest rate may affect optimal consumption through intertemporal elasticity of substitution. Simultaneous effects of the pure preference parameters, interest rates variations and the liquidity constraints on optimal consumption path are mathematically formulated. Limitations in optimal control applications in modeling optimal consumption with liquidity constraints in a stochastic environment are briefly examined.

---

[*] Associate Professor, Faculty of Economics, Allameh Tabatabaei University, email. Safipour1360@gmail.com

1. Introduction
Empirical work using both micro and macro data have shown that consumption is excessively sensitive to current income than what is warranted by permanent income-life cycle hypothesis.[1] The excess sensitivity of consumption to income can be attributed, among other things, to the lack of a perfect capital market. The fundamental assumption of LC-PIH is that households maximise their lifetime utility functions subject to their lifetime budget constraints when free borrowing and lending to smooth consumption are possible. Imperfect capital markets are largely characterized by borrowing or liquidity constraints. A "liquidity constrained" household cannot borrow freely to smooth its consumption trajectory over time, thus current income becomes a major determinant of current consumption.

The importance and consequences of liquidity constraints in models of consumption behaviour are discussed in section 2. Optimal control of multi-stage dynamic model of consumption behaviour under liquidity constraints is the main focus of the remaining four sections in this paper.

A rational forward-looking consumer is assumed to behave according to a familiar Ramsey model with additively separable utility function. Having a span of life T, the consumer is assumed to have an initial financial wealth A_0 and receives a real disposable income Y_t in period t. He consumes C_t in period t and it is further assumed that his rate of time preference is δ. Although the assumption of a known real rate of return is rather binding, it is used here for mathematical tractability. The assumption of an additively separable utility function is also a restriction on consumer's preferences, but it is commonly used in the literature because of its analytical convenience. It is further assumed that there is no rationing in the goods market, so consumers are not constrained by the purchase of goods and services.

The problem facing the consumer is to find the optimal consumption path which maximises his expected lifetime utility function subject to a lifetime budget constraint and an additional constraint on borrowing. This leads to the following maximization problem in standard calculus of variation,

---

1. For the early work on this subject , see, for example, Hall (1978), Hall and Mishkin (1982), Flavin (1985), Zeldes (1989), Cushing (1992) and Jappelli and Pagano (1994).

$$(1) \qquad J = \mathrm{Max}E \int_{t_0}^{T} e^{-\delta t} U\big[C(t)\big] dt,$$

subject to

$$(2) \qquad A_{t+1} = (1+r)A_t + Y_t - C_t,$$

$$(3) \qquad A_t \geq 0.$$

It is assumed that $U\big[C(t)\big]$ is increasing and strictly concave. Inequality (3) implies the existence of liquidity constraint. This means the consumer's end-of-period financial asset, after receiving income and allowing for consumption expenditure, cannot be negative. In other words, this inequality reflects the fact that consumers cannot consume today the income which they receive tomorrow. This condition can, of course, be generalised to

$$(4) \qquad A_t \geq -a,$$

where a is the limit on net indebtedness. Further generalization will also be considered, for example, in equation (12).

2. The Importance and Implications of Liquidity Constraints in Consumption Models

Liquidity constraints, as an explanatory variable in consumption models, necessarily introduce a number of interesting theoretical problems such as follows.

1. Liquidity constraints which represent the lack of *financial deepening,* can be regarded as an important determinant in consumption-saving behaviour in developing countries which are usually characterised by financial underdevelopment.

Defining financial deepening as the increases in the ratio of financial assets to GDP and defining *excess sensitivity parameter* as the fraction of consumers who are more sensitive to current income than what PIH implies, one expects the existence of an inverse relationship between excess sensitivity parameter and financial deepening. This leads to the hypothesis that McKinnon's type of

financial liberalisation[1] in developing countries, which would ease borrowing constraints, will reduce the exess sensitivity parameter. In this regard, one can argue that, in general, liquidity constraints make fiscal policies such as tax cuts or debt-financed fiscal spending more effective. A fall in current income affects consumption behaviour in developing countries more severely as compared to developed countries because a large portion of consumers in developing countries are liquidity constrained.[2]

2. The relationship between liquidity constraints and the aggregate saving rate is interestingly complex. The inability of households to borrow the desired amount in an imperfect credit market might lead to higher saving rates as compared with saving ratios in developed financial markets. This problem has received considerable attention in the literature on consumption-saving behaviour. Let us briefly refer to the early contributions. Hayashi, Ito and Slemrod (1988) have examined this property for the United States and Japan; Jappelli and Pagano (1994) have attributed the high saving rates in Italy to its relatively underdeveloped consumer credit and mortgage markets; Muellbauer and Murphy (1990) and Beyoumi (1991) argue that the sharp decline of the UK saving rates in the 1980s might be due to financial deregulations. By increasing saving rates, liquidity constraints might induce capital accumulation and hence can stimulate higher rates of growth.

The above conclusion might appear to be inconsistent with the McKinnon type argument that financial developments enhance the process of economic growth. McKinnon (1973) argues that by removing credit rationing, the resulting competitive financial intermediation promotes more efficient allocation of credit to investment and thus higher rates of return on capital can be achieved.[3] To reconcile the role of liquidity constraints in promoting growth rates with the McKinnon-Shaw model of financial liberalisation, one has to differentiate between credits to firms and credits to households [Jappelli and Pagano (1994)]. Such differentiation can be rationalised in view of the average loan size, informational asymmetries and the cost of contract enforcement. However, the argument that "if banks ra-

---

1. See McKinnon (1973)
2. See, for example, Hubbard and Judd(1986) and Heller and Starr(1979)
3. See McKinnon(1973). See, also Fry (1984), Bencivenga and Smith (1991) and Greenwood and Jovanovic (1990) among others.

tion credits to households while making it available to firms efficiently, capital accumulation and growth will be enhanced"[1] might be valid only within a static framework. The maximising behaviour of firms will become adversely affected by the behaviour of liquidity constrained households in a dynamic optimization context through the resulting changes in households' demand.

3. Demographic structure of the population is likely to be a key factor in explaining the relationship between liquidity constraints and saving rates. Faster growth rates might stimulate the consumption of the young and thus reduce saving rates.

Liquidity constraints usually apply more severely to the young portion of population. The young usually find liquidity constraints more binding in smoothing the consumption pattern over time. This is a point of particular importance in modelling the nature and effects of liquidity constraints in developing countries due to their higher proportions of young population.

The existence of young population and the pervasive liquidity constraints in developing countries make the Keynesian type consumption function more data admissible. This does not, however, reduce the importance of life-cycle pattern in consumption-saving behaviour in these countries because savings during the middle years would still be optimal if consumers wish to enjoy the period of retirement. This explains the fact that fiscal policies in developing countries which affect the current income is more effective to influence the consumption-saving trajectory. Moreover, as Hubbard and Judd (1986) argue, it is the young segment of the population which strongly feel the implications of such current income fluctuations.

4. Liquidity constraints may become closely linked with the concept of informational asymmetries and the related notions of adverse selection and moral hazards. The reason for the banks' unwillingness to lend freely to households for consumption purposes might be the uncertainties about households' future income as well as the risk of default. That explains why borrowing against the purchase of durable goods is not so binding because durable goods can be used as collateral.

---

1. Jappalli and Pagano (1994), p. 84.

Modelling the uncertainties associated with household's future income has been an active research work in theoretical developments in consumption theory. By adding the assumption of rational expectations to the standard life-cycle-permanent income hypothesis, Hall (1978) made the first significant attempt in formulating the stochastic implications of income in a consumption function. The household's decision on how much to borrow and save is not usually independent of the uncertainties about future events. This makes the question of insurance a matter of crucial importance in explaining saving-consumption behaviour in developing countries. The absence of an efficient system of insurance might further promote the precautionary saving motives. It follows that, as Besley (1993) maintains, savings, credits and insurance in developing countries are closely related with one another and can best be analysed within a unified theoretical framework.

Development of small scale indigenous financial institutions operating in rural and urban areas can partially relax household's liquidity constraints arising from their future income uncertainties. Such financial institutions may successfully administer the optimum allocation of loanable funds. The accuracy of information about potential borrowers will minimise the risk of adverse selection and moral hazards associated with credit allocation.

5. Household's uncertainties about their future income, a typical characteristic of developing countries, stimulate precautionary motives to save. These motives interact with liquidity constraints because in an underdeveloped credit market where households are usually unable to borrow when times are bad there exists an incentive for higher savings in good times.

6. Although any relaxations of liquidity constraints through improvements in consumer loan markets permit an individual to increase his consumption, this incremental consumption should be paid back with interest during the consumer's life-cycle. Provided that the individual's real income does not grow enough, this interest payment will constrain individual's future consumption. Despite the fact that such a decline in future consumption might be consistent with individual's utility maximisations a tendency may exist for aggregate consumption to fall when the economy is not growing fast enough to compensate for the aggregate interest payments.

7. Assuming that liquidity constrained consumers generally are more sensitive to current income variations as compared to liquidity unconstrained individuals, and assuming further that liquidity constrained and unconstrained individuals are usually the monetary debtors and creditors, respectively, it follows that a significant increase in the real rate of interest might have income redistributional effect between borrowers and lenders. This will affect the level of aggregate consumption through the existing differences between marginal propensities to consume. Note, however, that in the absence of liquidity constraints, the effect of changes in the interest rate on consumption is usually expected to be minimal because the resulting intertemporal substitution and wealth effects work in opposite directions.

8. To the extent that the role of inflation on consumption behaviour is reduced to the effects of inflation-induced changes in interest rates on consumption, an increase in the rate of inflation may affect the liquidity constrained consumption through interest rate variations.

9. Provided that the nominal rate of interest and the nominal credit ceilings are fully adjusted to accommodate the inflation rate, consumption will remain unaffected because consumer's real wealth has not changed. If the credit limit for liquidity constrained consumers are not revised, they are forced in a position to reduce their consumption in proportion to any higher loan repayments resulting from the increased interest rates. To the extent that liquidity constrained individuals reduce their consumption, the aggregate consumption may decline following inflation.

10. Under the circumstances that an inflation rate does not change the nominal rate of interest, the resulting fall in the real interest rate implies a redistribution of income from liquidity unconstrained lenders to liquidity constrained borrowers. The net effect on consumption appears to be indeterminate not because substitution and income effects work in opposite directions, but because there exists uncertainties on the future rate of inflation which hinders liquidity constrained consumers to increase their consumption in the first instance. However, as inflation proceeds, liquidity constrained consumers can increase their consumption.

11. The composition of consumer's asset portfolio is also important because the higher the degree of asset's liquidity in consumer's portfolio the lower would be the liquidity constraints. The purchase of

illiquid physical assets (houses and lands) may affect the liquidity constrained consumption behaviour in the following ways: *i*) it reduces the portion of liquid assets in consumers' portfolio, hence increases the liquidity constraints, *ii*) it constitutes collateral for borrowing, hence decreases the future liquidity constraints. Within this context, the problem of credit rationing appears to be of prime importance.

3. Optimality Conditions for Consumption Path with Liquidity Constraints using the Method of Dynamic Programming

We show how the Lagrange multiplier measures the amount by which consumer's utility will change resulting from the relaxation of borrowing constraints. Recall the Euler equation for the optimal consumption without liquidity constraints.[1]

$$(5) \qquad U'(C_t) = \frac{1+r}{1+\delta} E U'(C_{t+1}).$$

Our objective in this section is to generalize this equation to the case where consumption is constrained by liquidities. Consider a consumer who wishes to maximise the following objective function

$$J = \text{Max} E\left[\sum_{t=0}^{T}(1+\delta)^{-t}U(C_t)\right],$$

subject to

$$A_{t+1} = (1+r)A_t + Y_t - C_t,$$

and the liquidity constraint

$$A_t \geq 0.$$

The functional recurrence equation[2] for this problem is

$$(6) \qquad V(A_t) = \max_{C_t}\left\{U(C_t) + (1+\delta)^{-1}EV(A_{t+1}) + \pi(A_{t+1})\right\},$$

---

1. See Appendix1.
2. See Appendix1 for the concept and derivation of functional recurrence equation in optimal consumption functions without liquidity constraint. A generalization to consumption functions with liquidity constraints is straightforward.

where π is the Lagrange multiplier for liquidity constraints. Note that the notation λ has been reserved as the multiplier for the equation of motion. Differentiate the right hand side with respect to C_t to obtain

$$\frac{\partial U(C_t)}{\partial C_t} + (1+\delta)^{-1} E\left\{\frac{\partial V(A_{t+1})}{\partial A_{t+1}} \cdot \frac{\partial A_{t+1}}{\partial C_t}\right\} + \frac{\partial \pi(A_{t+1})}{\partial A_{t+1}} \cdot \frac{\partial A_{t+1}}{\partial C_t} = 0,$$

or

(7) $$\frac{\partial U(C_t)}{\partial C_t} - (1+\delta)^{-1} E\left\{\frac{\partial V(A_{t+1})}{\partial A_{t+1}}\right\} - \pi = 0.$$

For the envelope relation, consider a small variation in A_t in equation

(6) $$V'(A_t) = (1+\delta)^{-1} E \frac{\partial}{\partial A_t} V(A_{t+1}) + \pi \frac{\partial(A_{t+1})}{\partial A_t},$$

or

(8) $$V'(A_t) = (1+\delta)^{-1}(1+r)EV'(A_{t+1}) + (1+r)\pi.$$

If we now transfer the second and the third terms in equation (7) to the right hand side and then multiply both sides by $(1+r)$ we obtain exactly the right hand side of equation (8). We can then write

$$V'(A_t) = (1+r)U'(C),$$

or

(9) $$V'(A_{t+1}) = (1+r)U'(C_{t+1}).$$

Substituting equation (9) into equation (7) gives

$$U'(C_t) = (1+\delta)^{-1}\left[(1+r)EU'(C_{t+1})\right] + \pi,$$

or

(10) $$U'(C_t) = \frac{1+r}{1+\delta} EU'(C_{t+1}) + \pi.$$

Equation (10), which is the generalization of equation (5), is the optimality condition for consumption behaviour of a consumer with liquidity constraints. π which is the Lagrangian multiplier associated with the liquidity constraint measures the amount by which consumer's utility will change if current constraints on borrowing become relaxed by one unit. In other words, π represents the amount by which the marginal utility of borrowing will increase at period t by reducing the consumption next period. If π becomes zero, the liquidity constraint will be totally relaxed and the optimality condition (10) will be the same as in equation (5) where the marginal rate of substitution equals the marginal rate of transformation.

4. The Generalized Hamiltonian, Liquidity Constraints and the Rejection of Hall's Random Walk Hypothesis

In section 3, we used Bellman's dynamic programming to derive the optimality conditions in consumption path when liquidity constraints were binding. In this section, we first obtain the same result, i.e. equation (10), by applying the generalised Hamiltonian function in the maximum principle and then, using Pontryagin maximum principle we demonstrate the rejection of Hall's random walk hypothesis when liquidity constraints bind.

Specifically, we answer the following two questions in this section: *i*) how can the existence of liquidity constraints directly influence the rate of change in the optimal consumption trajectory; and *ii*) how does the Hall's random walk hypothesis of consumption collapse when an individual consumer is facing with the liquidity constraint. These two questions and particularly the latter are of prime theoretical value since our results here constitute, for the first time, a strong theoretical framework for empirical tests because it explicitly formulates the effects of liquidity constraints on optimal consumption path.

We assume that consumers, in an imperfect capital market, face an upper limit to their net indebtedness which is a function of their income. The problem is to maximize the objective function, [equation (1)],

$$J = \mathrm{Max}E\int_{t_0}^{T} e^{-\delta t}U\big[C(t)\big]dt,$$

subject to the asset transition equation (2) which in control terminology is usually called the *equation of motion* or *system dynamics.* Equation (2) can be written as[1]

$$\dot{A}(t) = rA(t) + Y(t) - C(t),\qquad (11)$$

and the assumption of liquidity constraints,

$$A(t) \geq -a - bY(t),\qquad (12)$$

where a is the limit of net indebtedness.
Define the Hamiltonian equation,

$$H = EU[C(t)]e^{-\delta t} + \lambda(t)[rA(t) + Y(t) - C(t)],$$

where $\lambda(t)$ is the adjoint variable. The control variable, $C(t)$, should maximise H subject to the inequality constraint (12). Writing the inequality (12) as $A(t) + a + bY(t) \geq 0$, we can construct the generalised Hamiltonian as follows,

$$H^* = EU[C(t)]e^{-\delta t} + \lambda(t)[rA(t) + Y(t) - C(t)] + \pi(t)[A(t) + a + bY(t)].$$

The control variable $C(t)$ can maximise H^* if

$$\frac{\partial H^*}{\partial C(t)} = 0,$$

or

$$EU'[C(t)]e^{-\delta t} = \lambda(t).\qquad (13)$$

Equation (13) gives the optimal consumption $C(t)$ as a function of the adjoint variable $\lambda(t)$. To obtain the time path of the adjoint variable, we use the canonical equation, i.e.

$$-\frac{\partial H^*}{\partial A(t)} = \dot{\lambda}(t),$$

---

1. See Appendix 2

or

(14)
$$\dot{\lambda}(t) = -r\lambda(t) - \pi(t).$$

To obtain the properties of optimal consumption policy, we differentiate equation
(13) with respect to time. Since $U'[C(t)]$ is a function of $C(t)$, we have

$$EU''[C(t)]E\frac{dC(t)}{dt}e^{-\delta t} - \delta e^{-\delta t}EU'[C(t)] = \dot{\lambda}(t),$$

and by using the equation for $\dot{\lambda}(t)$ we obtain

$$\{EU''[C(t)]E\dot{C}(t) - \delta EU'[C(t)]\}e^{-\delta t} = -r\lambda(t) - \pi(t).$$

Substituting equation (13) into the right hand side of the above equation and dividing both sides by $e^{-\delta t}$, we have

$$EU''[C(t)]E\dot{C}(t) - \delta EU'[C(t)] = -rEU'[C(t)] - \pi(t)e^{\delta t},$$

or

$$\frac{EU''[C(t)]}{EU'[C(t)]}E\dot{C}(t) - \delta + r = -\frac{\pi(t)e^{\delta t}}{EU'[C(t)]}.$$

Assuming that

(15)
$$\psi(t) = \frac{\pi(t)e^{\delta t}}{EU'[C(t)]},$$

and noting that $\rho_a^* = -\dfrac{EU''[C(t)]}{EU'[C(t)]}$ is the coefficient of absolute risk aversion, we have,

(16)
$$\rho_a^* E\dot{C}(t) = r - \delta + \psi(t).$$

Equation (16), is an important result in consumption optimization when liquidity constraints bind. This equation implies that with a concave utility function, where $U''[C(t)]$ is negative and thus the coefficient of absolute risk aversion is positive, if liquidity constraints are binding at time t, i.e. $\psi(t) > 0$, then the optimal consumption increases, $\dot{C}(t) > 0$, provided that the interest rate is more than or equal to the subjective rate of time preferences, i.e. $r \geq \delta$. If liquidity constraints are not binding, $\psi(t) = 0$, then consumption will increase if $r > \delta$. However, equation (16) implies that with a concave utility function, the existence of liquidity constraints makes the optimal consumption to grow, not only when the interest rate is more than or equal to the subjective rate of time preference ($r \geq \delta$), but even when $r < \delta$, provided that $\psi(t) > r - \delta$. Under such conditions one may conclude that the liquidity constraint may shift the optimal consumption profile forward even when the rate of time preference exceeds the interest rate.

We now prove that the existence of liquidity constraints invalidates the Hall's ran- dom walk hypothesis of optimal consumption. Recall that the Hall's hypothesis has been frequently tested for the explanatory power of variables (other than consumption) in predicting consumption for the next period. Hall's hypothesis has been rejected since variables such as lagged stock prices or lagged income proved to be significant in explaining current consumption [see, for example, Hall and Mishkin(1982) and Zeldes (1989)]. It is well-known that the failure of random walk hypothesis of consumption is usually attributed to the presence of liquidity constraints, thus the existence of liquidity constraints and its impact on con- sumption have been tested indirectly. I have been unable to find, in the published literature on this subject, a complete theoretical treatment which shows *i*) how the existence of liquidity constraints can directly affect the optimal path of con- sumption and *ii*) how the random walk hypothesis of consumption breaks down when liquidity constraints are binding. I have answered the former question by deriving equation (16) and will answer the latter as follows.

Using the mean value theorem, we can write equation (16) as

$$E[C(t+h) - C(t)] = \frac{h[r - \delta + \psi(t)]}{\rho_a^*}.$$

By assuming $h = 1$, we have

(17)
$$C(t+1) = \frac{1}{\rho_a^*}(r - \delta) + C(t) + \frac{1}{\rho_a^*}\psi(t) + \xi_{t+1}.$$

Using Hall's condition for the random walk hypothesis, i.e. $r = \delta$, we have

(18)
$$C(t+1) = C(t) + \frac{1}{\rho_a^*}\psi(t) + \xi_{t+1},$$

where

$$\psi(t) = \frac{\pi(t)e^{\delta t}}{EU'[C(t)]},$$

is defined in equation (15) and $\pi(t)$ is the adjoint variable associated with liquidity constraint in the generalised Hamiltonian function. Equation (17) implies that with a concave utility function, the expected value of optimal consumption increases when *i)* $r \geq \delta$ and *ii)* $\psi(t) > r - \delta$ even if $r < \delta$. Moreover, equation (18) indicates that the Hall's random walk hypothesis can be rejected if liquidity constraints bind, i.e. $\psi(t) \neq 0$. This result holds even under the condition $r = \delta$.

5. Time-varying Interest Rates and the Properties of Optimal Consumption Path under Liquidity Constraints

As discussed before, the Euler equation approach in modelling consumer behaviour, which has been initiated by Lucas (1976) in his critique of standard estimation of consumption function, is based on the first order conditions in an individual's intertemporal optimization problem [equation (5)]. We have also noted that the rejection of the Euler equation[1] has been related to the existence of liquidity

---

1. See, for example, Flavin(1982, 1985), Muellbauer (1983), Mankiw, Rotemberg, Summers (1985) and Campbell and Mankiw (1989) for the early work on this subject.

constraints.[1] Note that the tests for the existence of liquidity constraints are usually carried out either indirectly, or are simply based on a particular assumed consumption-income relationship. Such problems in modelling the existence of liquidity constraints, within a rational expectations life-cycle-permanent-income hypothesis, are mainly rooted in the difficulties associated with finding satisfactory proxies for liquidity constraints. King (1986), Hayashi (1987) and Muellbauer and Latimore (1995) are among the earlier work which have reported the achievements in modelling the liquidity constraints in this direction.

It is possible to relax this theoretical shortcoming substantially by, first, introducing a function representing the structure of liquidity constraints (or the nature of capital market imperfections) and then accommodating this function within an individual's intertemporal optimization problem. This approach, which was adopted in the previous section, will be further developed here. We show how the generalised Hamiltonian function can be useful in modelling this problem.

In section 5.1 we will obtain an equation similar to equation (16) in which the change in consumption is related to the liquidity constraint, interest rates and consumer's time preferences. However, time-varying interest rates do not change our general conclusion which states that the existence of liquidity constraints necessarily results in an increasing consumption over time if the interest rate becomes equal to the subjective rate of time preferences. However, when $r(t) < \delta$, the consumption might not increase even if liquidity constraints bind. This, of course, depends entirely on the severity of the liquidity constraints: the condition $\psi(t) > r(t) - \delta$ might ensure a forward shift of optimal consumption profile when consumer's time preferences exceed the rate of interest. Using the Kuhn-Tucker conditions, we will analyse, in section 5.2, the interactions between time-varying interest rates, the utility discount rate and the severity of liquidity constraints. This section generalises the results of Heller and Starr (1979) to the case where the interest rate is time-varying and the liquidity constraint specifies the consumer's net indebtedness as a function of income.

---

1. See, for example, Hall and Mishkin (1982), Hayashi (1985) and Zeldes (1989) for the early work on this subject.

Using an inverse relationship between the coefficient of absolute risk aversion and the intertemporal elasticity of substitution, we obtain, in section 5.3, a relationship in which the intertemporal elasticity of substitution appears as the coefficient of liquidity constraint. This produces the important result that the re- sponse of optimal consumption to variations in the severity of liquidity constraints will be conditioned by the consumer's intertemporal elasticity of substitution.

5.1. Time-varying Interest Rates and Liquidity Constraints
This section generalises the results we obtained in section 4 by using a time-varying interest rate. In equation (16), we demonstrated the effects of liquidity constraints on the optimal time path of consumption, and equation (17) clearly showed how the existence of liquidity constraints could invalidate Hall's random walk hypothesis of consumption. We will examine these results under the condition of time-varying interest rates. The generalised Hamiltonian function will be used throughout.
The problem is to maximise the following objective function,

$$J = \mathrm{Max} E \int_{t_0}^{T} e^{-\delta t} U[C(t)] dt,$$

subject to the asset transition equation,

(19) $$\dot{A}(t) = r(t)A(t) + Y(t) - C(t),$$

and the following constraints on borrowing, i.e., equation (12),

$$A(t) \geq -a - bY(t).$$

Note that the specification of the objective function and liquidity constraints are as before whereas the asset transition equation embodies a time-varying interest rate.
Defining the generalised Hamiltonian as

(20) $$\begin{aligned} H^* = EU[C(t)]e^{-\delta t} + \lambda(t)[r(t)A(t) + Y(t) - C(t)] \\ + \pi(t)[A(t) + a + bY(t)], \end{aligned}$$

the optimal value of consumption will maximise H^* if

$$\frac{\partial H^*}{\partial C(t)} = 0,$$

or

(21)
$$EU'[C(t)]e^{-\delta t} = \lambda(t).$$

To obtain the time path of adjoint variable $\lambda(t)$, we use the canonical equation

$$-\frac{\partial H^*}{\partial A(t)} = \dot{\lambda}(t),$$

or

(22)
$$\dot{\lambda}(t) = -r(t)\lambda(t) - \pi(t).$$

Differentiating equation (21) with respect to time and substituting for $\dot{\lambda}(t)$ from equation (22) yields

$$EU''[C(t)]E\dot{C}(t) - \delta EU'[C(t)] = -r(t)EU'[C(t)] - \pi(t)e^{\delta t}.$$

Dividing both sides by $EU'[C(t)]$ and defining $\psi(t)$ as

$$\psi(t) = \frac{\pi(t)e^{\delta t}}{EU'[C(t)]},$$

yields

$$E\rho_a E\dot{C}(t) = r(t) - \delta + \psi(t),$$

or

(23)
$$\rho^* E\dot{C}(t) = r(t) - \delta + \psi(t),$$

where ρ_a is the coefficient of absolute risk aversion and $\rho_a^* = E\rho_a$. Equation (23) is the generalization of equation (16), and it is assumed that the interest rate is not constant. According to equation (23), for

any concave utility function, where $\rho_a > 0$, the existence of liquidity constraints, $\psi(t) > 0$, does not necessarily result in an increasing consumption over time. This is due to the variability of the interest rate. Despite structural similarities between equations (23) and (16), one can argue that for periods in which $r(t) < \delta$, consumption might not increase due to the existence of liquidity constraints if $\psi(t) < \delta - r(t)$. Moreover, the condition $\psi(t) > r(t) - \delta$ ensures that optimal consumption profile will shift forward even if $r(t) < \delta$.

5.2. Liquidity Constraints and the Interactions between r(t) and δ

In the previous section we proved that the effect of liquidity constraints on the pattern of optimal consumption over time depends on the relative magnitude of time varying rates of interest, the utility discount rate and the severity of liquidity constraints. In this section, we show that the Kuhn-Tucker conditions for optimization of consumer's utility over time can provide a useful relationship between these factors. Heller and Starr (1979) have used the Lagrangian approach to the same problem with time-invariant interest rates and a liquidity constraint in the form of non-negative assets, i.e. $A(t) \geq 0$. This section extends their analytical framework and generalises their results by introducing *i*) time-varying interest rates and *ii*) a liquidity constraint which specifies consumer's net indebtedness as a function of his income.

Consider an individual consumer maximising the following discrete utility function,

$$(24) \qquad J = \text{Max} \sum_{t=0}^{T} (1+\delta)^{-t} U(C_t),$$

subject to the following asset transition equation

$$(25) \qquad A_{t+1} = (1+r_t)A_t + Y_t - C_t,$$

and the liquidity constraints,

$$(26) \qquad A_t \geq -a - bY_t.$$

It is assumed that the consumer's initial asset is non-negative and is given, i.e.

$$A_0^* \geq 0.$$

The Lagrangian function for this problem is

(27) $\quad L = \sum_{t=0}^{T} \left\{ (1+\delta)^{-t} U(C_t) + \lambda_t \left[(1+r_t)A_t + Y_t - C_t - A_{t+1} \right] + \pi_t \left[A_t + a + bY_t \right] \right\}$

The second term in equation (27) includes A_t and A_{t+1}. To avoid problems which might arise in differentiating L with respect to A_t, we can divide the planning horizon in the Lagrangian as follows,

$$L = U(C_0) + \lambda_0 \left[(1+r_0)A_0^* + Y_0 - C_0 \right] + \pi_0 \left[A_0^* + a + bY_0 \right] +$$

$$\sum_{t=1}^{T} \left\{ (1+\delta)^{-t} U(C_t) + \lambda_t (Y_t - C_t) + \left[(1+r_t)\lambda_t - \lambda_{t-1} \right] A_t + \pi_t \left[A_t + a + bY_t \right] \right\}$$

The Kuhn- Tucker necessary conditions for an optimum are

(28) $\quad \dfrac{\partial L(C_t, A_t, r_t, \lambda_t, \pi_t)}{\partial C_t} \leq 0, \quad C_t \geq 0 \;$ *with complementary slackness,*

(29) $\quad \dfrac{\partial L}{\partial A_t} = 0,$

(30) $\quad \dfrac{\partial L}{\partial \lambda_t} \geq 0, \quad \lambda_t \geq 0 \qquad$ *with complementary slackness,*

(31) $\quad \dfrac{\partial L}{\partial \pi_t} \geq 0, \quad \pi_t \geq 0 \qquad$ *with complementary slackness.*

Assuming $U'(0) = +\infty$, the condition $C_t > 0$ in equation (28) ensures that the left hand inequality binds, i.e.

(32) $\qquad\qquad\qquad (1+\delta)^{-t} U'(C_t) - \lambda_t = 0.$

From equation (29) we have

$$(1+r_t)\lambda_t - \lambda_{t-1} + \pi_t = 0,$$

or

(33) $$\lambda_{t-1} - (1+r_t)\lambda_t = \pi_t.$$

In equation (30), $\dfrac{\partial L}{\partial \lambda}$ gives the asset transition equation which is an equality by definition. Therefore, $\dfrac{\partial L}{\partial \lambda}$ is binding and thus $\lambda(t)$ should be slack, i.e.

(34) $$\lambda_t \geq 0.$$

By the same argument, equation (31) gives

$$A_t + a + bY_t \geq 0, \quad \pi_t \geq 0 \qquad \text{with complementary slackness.}$$

If our liquidity constraints bind, then we have

(35) $$\pi_t \geq 0,$$

or

(36) $$\pi_t(A_t + a + bY_t) = 0$$

Equation (36) implies that the Kuhn- Tucker multiplier for liquidity constraint is non-zero if equation (26) holds, i.e. liquidity constraints bind. By equation (33) we have

(37) $$\lambda_{t-1} \geq (1+r_t)\lambda_t,$$

if π_t is nonnegative. According to equation (36), if liquidity constraints bind then (37) holds with strict inequality. From equation (32) we have

$$(1+\delta)^{-(t-1)} U'(C_{t-1}) = \lambda_{t-1},$$

or alternatively,

(38)
$$U'(C_{t-1}) = (1+\delta)^{t-1}\lambda_{t-1},$$

which by substituting equation (37) for λ_{t-1} yields

(39)
$$U'(C_{t-1}) > (1+\delta)^{t-1}(1+r_t)\lambda_t.$$

From equation (32) we know that

(40)
$$U'(C_t) = (1+\delta)^t \lambda_t.$$

Substitute equation (40) into equation (39) to obtain

$$U'(C_{t-1}) > U'(C_t)(1+\delta)^{-1}(1+r_t),$$

or

(41)
$$U'(C_{t-1}) > \frac{1+r_t}{1+\delta}U'(C_t).$$

Equation (41), which is based on the assumption of binding liquidity constraint, is an important result. It clearly invalidates Hall's random walk hypothesis.

Recall that Hall's random walk hypothesis of consumption is based on the assumption of equality between interest rate and subjective time rate of discount.

Equation (41) implies that even if $r_t = \delta$, we have

(42)
$$U'(C_{t-1}) > U'(C_t).$$

With regard to the assumed concavity of utility function, i.e. $U''(C_t) < 0$, equation (42) implies that consumption is increasing over time whenever the liquidity constraint is binding.

5.3. Interest Rates, Intertemporal Elasticity of Substitution, and Liquidity Constraints

It is well known that the effect of interest rate variations on consumption can best be analysed by intertemporal elasticity of substitution. I will demonstrate how the optimal control theory can

contribute towards formulating this problem by utilising the coefficient of absolute risk aversion.

Let us now present our formulation of interactions between interest rates, intertemporal elasticity of substitution and liquidity constrained consumption. Intuitively, the intertemporal elasticity of substitution can be negatively related to the coefficient of absolute risk aversion. Defining

$$\rho_a = -\frac{U''[C(t)]}{U'[C(t)]} = -\frac{d\ln U'[C(t)]}{dC}$$

as the coefficient of absolute risk aversion (*CARA*), we know that $U''[C(t)]$ is a measure of concavity of the utility function. It is known that a consumer with a sharply concave utility function will, according to *LC-PIH*, avoid intertemporal substitution and will, therefore, prefer to smooth the consumption path over his planning horizon. Since $U''[C(t)] < 0$, such consumers will have high ρ_a which accompanies a low intertemporal elasticity of substitution. The same argument applies for constant relative risk aversion ρ_r defined as [see Selden (1978)],

$$\rho_r = -\frac{U''[C(t)]}{U'[C(t)]}.C(t) = -\frac{d\ln U'[C(t)]}{d\ln C(t)}.$$

It should be noted that only for some class of utility functions is the intertemporal elasticity of substitution just equal to the reciprocal of the coefficient of absolute risk aversion [see Hall (1985)]. However as Hall (1988) reports, it seems that the best way to estimate the intertemporal elasticity of substitution is simply by regressing the log-change in consumption on the expected real interest rate because, intuitively, the rate of change in consumption over time can reveal the magnitude of the intertemporal elasticity of substitution in consumption.

In what follows, I derive an approximate relation between the intertemporal elasticity of substitution and the coefficient of absolute risk aversion. Using this relationship, I will then show how the time-varying interest rate and liquidity constraint can affect the changes in optimal consumption through the intertemporal elasticity of

substitution. More specifically, we demonstrate that the consumer's intertemporal elasticity of substitution will condition the response of optimal con- sumption variations to liquidity constraints.

Defining the intertemporal elasticity of substitution as

$$(43) \qquad \sigma = \partial \ln \left\{ \frac{U'[C(t+1)]}{U'[C(t)]} \right\} / \partial \ln r_t ,$$

and expanding $U'[C(t+1)]$ by a Taylor series, we have

$$U'[C(t+1)] = U'[C(t)] + \Delta C(t+1) U''[C(t)] + \cdots .$$

Dividing both sides by $U'[C(t)]$ yields

$$(44) \qquad \frac{U'[C(t+1)]}{U'[C(t)]} = 1 + \Delta C(t+1) \frac{U''[C(t)]}{U'[C(t)]} .$$

Note that $\dfrac{U''[C(t)]}{U'[C(t)]}$ in the second term on the right hand side of equation (44) is exactly ρ_a with an opposite sign. Thus ρ_a will be negatively related to $\dfrac{U'[C(t+1)]}{U'[C(t)]}$ on the left hand side of this equation. This completes our proposition that the coefficient of absolute risk aversion and intertemporal elasticity of substitution are inversely related, i.e.

$$(45) \qquad \rho_a = f^{-1}(\sigma).$$

We can now return to our familiar consumption optimization problem with a liquidity constraint of the form $A(t) \ge -a - bY(t)$ and an asset transition equation with time-varying interest rate. Recall equation (23), which was obtained by the application of Pontryagin's maximum principle, namely,

$$E \rho_a E \dot{C}(t) = r(t) - \delta + \psi(t),$$

where $\psi(t)$ was defined as

$$\psi(t) = \frac{\pi(t)\, e^{\delta t}}{EU'\big[C(t)\big]},$$

and $\pi(t)$ is the adjoint variable associated with the liquidity constraint. Substi- tuting equation (45) into equation (23) gives

(46) $$E\dot{C}(t) = f(\sigma)r(t) - f(\sigma)\delta + f(\sigma)\psi(t).$$

Equation (46) has important implications. It clearly specifies how the rate of interest affects the change in consumer's optimal consumption trajectory through the intertemporal elasticity of substitution as its coefficient. Moreover, it states that the intertemporal elasticity of substitution appears as the coefficient of the liquidity constraint. Equation (46) implies that the response of optimal consumption to liquidity constraints will be conditioned by the consumer's intertemporal elasticity of substitution. It is interesting to examine the term $f(\sigma)\psi(t)$. This term captures simultaneously the effects of *i*) pure preference parameters such as the utility function and the subjective rate of preference; *ii*) interest rate variations; and *iii*) structural parameters in the credit market which are manifested in the formulation of liquidity constraints. The first two factors are reflected by the intertemporal elasticity of substitution and the third factor is captured by $\psi(t)$. Moreover, according to equation (46), the time-varying interest rate affects the relationship between liquidity constraints and optimal consumption policies through its effects on the intertemporal elasticity of substitution.

6. Considerations on Optimal Consumption in a Stochastic Environment: The Basic Shortcomings of Optimal Control Approach

The discussion we had so far on deterministic dynamic choice-theoretic consumption models was based on the following assumptions: 1) The utility function, representing preferences for the objective of choice, is monotonically increasing and strictly concave; 2) Constraints facing the agent can be summarised in a budget constraint; and 3) The agent's optimal choice maximises utility over the planning horizon subject to the assumed constraints. We have also used the following auxiliary assumptions: 1) The agent is assumed to

be rational; 2) The agent's planning horizon is the lifetime of himself (and his spouse if applicable and not of his parents or mature children); 3) The agent has direct access to perfect capital and insurance markets; 4) The agent must be solvent at the end of the planning period; and 5) The intertemporal utility function is additive and depends on real consumption expenditures in each period. Moreover, it is assumed that the following variables are given: the length of the agent's lifetime or the planning horizon, the rate of return on investment, and the non-interest income which is also assumed to be exogenous. Different variants of the standard choice-theoretic consumption models are possible. An example is the existence of borrowing constraints which yields the liquidity constrained consumption models discussed in sections 4 and 5.

Within the above context, uncertain lifetime and uncertain future income are the two major sources which give rise to *stochastic optimal consumption behaviour.* Despite the importance of uncertain labour income in deciding on optimal consumption plans as well as a great deal of research work on life cycle and permanent income hypotheses, "yet, closed form decision rules for optimal consumption in the presence of uncertain labour income have not, in general, been derived. It seems strange that so much theoretical and empirical work has been done studying consumption and yet we do not even know what the optimal level of consumption or sensitivity of consumption to income should be in most very simple settings" (Zeldes, 1989, p. 275).

Considerations regarding the failure of Euler approach, from a control theoretic point of view, in explaining optimal consumption behavior can be summarized as follows:

1. Uncertain lifetimes can partially be responsible for uncertainty in future non-interest income simply because the latter is contingent on the survival of the agent and, therefore, becomes uncertain.

2. The assumption of dependency of utility derived from consumption on health invalidates the simple structure of the objective function defined earlier.

3. For a multiple-person family, the uncertain lifetime renders the maximisation of the objective function subject to the asset transition equation and the liquidity constraints, inexpressible in a deterministic form presented in sections 4 and 5.

4. Multiple-person families pose serious problems for optimal consumption plans when lifetimes are uncertain. Optimal consumption in each period is contingent on family consumption in that period as well as on the probability distribution of future family consumption. The complicated mathematical expectations of family compositions at different periods make it difficult to apply the method of dynamic programming to derive closed form solutions for optimal consumption decisions. An alternative approach is to assume that the family behaves as if expected family consumption in each future period will be realized with certainty. Mariger (1986) has used this approach to model the econometric specification of optimal consumption behaviour with uncertain lifetimes. The difficulty is to revise, at each period, the expectations of family composition in future periods in order to reflect the new information which has become available to the family. It follows that the optimal family's consumption plan cannot be projected without the knowledge of the time-paths of family composition since the optimal consumption plan is contingent on this composition.

5. It was usually agreed that "it is not possible to obtain a closed-form solution for the optimal consumption plan when future labour income is uncertain" (Mariger, 1986, p. 59). The usual remedy was to eliminate all relevant income uncertainties by assuming a full insurance for net labour income in each period provided that at least one family member is alive. Note that there is no incentive for a single-person family to purchase such insurance.

Capital income uncertainty, or more specifically, rate of return uncertainty, poses serious problems for dynamic optimal consumption decisions. The standard procedure is to take expectations over the portfolio rate of return with risky assets. The difficulty arises because the individual consumer must evaluate, in each period, the likelihood of becoming liquidity constrained in future periods when deciding on optimal current consumption. Simplifying assumptions can, of course, help towards obtaining a tractable solution. For example, it is usually assumed that the joint distribution of asset returns in each future period is known in the initial stage.[1]

Hakansson (1970), as the pioneer worker in this field, considers an agent facing risky assets whose rate of return are independently and

---

[1]. See Hakansson (1970) and Merton (1971) for the early seminal work on optimal investment and consumption strategies under risk.

identically distributed over time. He assumes that the intertemporal utility function is additive and is of the isoelastic form. Merton (1971), as another pioneer researcher in this field, essentially considers the same problem for the continuous case where he assumes a Weiner process for asset returns. The assumption that the joint distribution of asset return in each period are known in the initial period ensures that all the relevant past information is reflected in the current level of net worth. Hence, future wealth is the only uncertain variable in the model which is relevant to future consumption. Optimal consumption plan can thus be reduced to a consumption-investment plan contingent on wealth in each period. Note that the assumption of independently distributed capital returns plays a key role in this dynamic optimisation, since, otherwise, optimal consumption in each period would be contingent not only on wealth but on the sequence of past related rates of return on risky assets.

An important finding of Hakansson and Merton is that the demand for risky assets in each period is proportional to wealth, the proportion depending only on the joint distribution of asset returns. Moreover, the increased *capital risk* decreases (increases) the propensity to consume wealth in each period if η is less (greater) than zero, and η is the exponent on consumption in an isoelastic utility function (used in Hakansson and Merton work) which measures the degree of concavity of the intertemporal utility function. A large value of η implies a larger consumption growth rate if the rate of interest exceeds the coefficient of time preference. It should be noted that according to Rothschild and Stiglitz (1971), a capital asset is riskier if the uncertain rate of return, r, is augmented to $r + \varepsilon$, where ε is distributed as white noise.

The studies mentioned above, did not, unfortunately, pay much attention to the important role of liquidity constraints in their analysis of the impact of stochastic rate of return on optimal consumption plans. Recall that their major findings imply that the proportion of wealth invested in risky assets depends *only* on the joint distribution of asset return, that is, it is independent of the size and the composition of agent's wealth. It follows, therefore, that the individual consumer should borrow heavily in the early phases of the life cycle when his full wealth is mainly in the form of future earnings. Hence, an optimal portfolio choice can only be modelled correctly when liquidity constraints are fully taken into account. The converse is also

true: an optimal consumption plan which takes liquidity constraints into account but does not acknowledge the effects of portfolio choice on risky assets will be equally inaccurate.

The findings of Hakansson and Merton, initially developed for a two-period model, were then generalised by Sibley (1975) and Miller (1976) for a multi-period optimal consumption plan with uncertain income. They showed that the results obtained by Hakansson and Merton are true for multi-period models. They also showed that beginning at any initial level of uncertainty on income, the precautionary saving increases with a Rothschild-Stiglitz (1970) mean-preserving spread on income. These results were further generalised by Zeldes (1984) to account for the sensitivity of consumption to wealth or to transitory income, i.e. the slope of the consumption function rather than its level. Using a second-order Taylor expansion of marginal utility, Zeldes showed that with constant relative risk aversion adding uncertainty raises the slope of the consumption function. This implies that income uncertainty makes consumption more sensitive to transitory income than under certainty equivalence. An interesting related result is that with constant absolute risk aversion the consumption function would shift downward in a parallel way when uncertainty is added, leaving the slope unchanged. In a similar development, Roel (1986) and Kimball (1988) have shown that excess sensitivity will occur for a class of utility functions that include constant relative risk aversion and excludes constant absolute risk aversion.[1]

6. Modelling stochastic future income in an individual's optimal consumption behaviour has always been a challenging issue. The early attempts in this field assumed that the individual consumer behaves *as if* expected future income is certain (see, for example, Hall and Mishkin, 1982). However, the seminal work of Dreze and Modigliani (1972) can be regarded as the most influential early contribution on optimal consumption decisions under uncertainty. It should be noted that Dreze and Modigliani's work benefited from the earlier work of Leland (1968) and Sandmo (1970) on optimal saving decisions under uncertainty.

The early works on closed-form solutions for dynamic optimal consumption plans with stochastic income and constant absolute risk

---

1. See also Deaton (1991), Carroll (1992, 1994) and Flavin (1993).

aversion have been reported by Schechtman and Escudero (1977), Caballero (1987) and Kimball and Mankiw (1987), among others. Zeldes (1989)] is the first author who has successfully formulated an exact closed-loop solution for optimal consumption with uncertain income and a utility function which is constant relative risk aversion. Following Hall and Mishkin (1982), Zeldes assumes that income can be decomposed into two separate components. The first is the "lifetime" or permanent component which is assumed to follow a geometric random walk and is disturbed at each period by a random shock which captures the effects of pay rises, job changes, health changes and other similar persistent factors. The other component is the transitory component which is assumed to follow an *MA*(*2*) and is hit each period by a random shock representing the effects of one-time bonuses, unemployment spells and other similar transitory factors. It is assumed that these two components are separately observable.

Zeldes (1989) has, for the first time, used the stochastic dynamic programming to calculate the optimal consumption plan with uncertain income. He formulated the problem simply as a stochastic control problem with only one state variable (wealth), one control variable (consumption) and one disturbance variable (income). Using a technique of Bertsekas (1976) in stochastic dynamic programming, Zeldes discretized the state space into an *S* element grid. Beginning from the terminal period, a backward induction is used to solve for the value function and the cor- responding optimal consumption. At each stage, the sum of current utility and the discounted expected value of next period's value function was maximised to yield the optimal level of consumption. It is well-known that the accuracy of the results depends upon the width of the grid used for the discretization in the stochastic dynamic programming framework. The approximate errors can thus be made arbitrarily small by narrowing the width of the grid at the expense of more computing time and excessive computer memory known as the *curse of dimensionality* in Bellman's dynamic programming.

However, Zeldes (1989) reports that the resulting consumption function is quite different from the certainty equivalence benchmark. The rational individuals with constant relative risk aversion develop an optimal consumption plan which "exhibits *excess* sensitivity to transitory income, hence they save *too* much and have expected growth of consumption that is *too* high relative to the simple

permanent income hypothesis benchmark even in the absence of borrowing constraints" (pp. 295-296).

7. Assuming that an individual consumer is facing an uncertain future income poses serious problems in the computational aspects of optimal control applications to consumption optimisation. Using Bellman's dynamic programming, the numerical complexities in calculating optimal consumption path when future income in uncertain, may render the standard optimisation of rational expectations permanent income/life cycle hypotheses unacceptable models of individual's consumption behaviour. However, the application of alternative optimal control techniques, i.e. Pontryagin's maximum principle, is not promising either since obtaining a closed-form solution, when income is stochastic, requires complicated and heavily involved iteration techniques.

We are, therefore, faced with a very serious methodological problem in the formulation of dynamic intertemporal consumption decisions with stochastic income. Computational complexities inherent in the method of stochastic dynamic programming and the maximum principle play the key role in this problem. The basis of the argument, which also constitutes a criterion for assessing the usefulness of an optimisation procedure, is best explained by Pemberton (1993, p. 3): "For the optimal solution to a model to be a useful guide to actual behaviour requires that the relevant agents in the real world can themselves identify and attain the solution (though not necessarily by using the same methods)". On the basis of this criterion, the standard stochastic dynamic programming and the maximum principle fail to be accepted as efficient and practical optimisation methods in dynamic optimal consumption decisions because of the inherent excessive backward inductive procedures. In fact, "either the problem gets so hideously complex that it is beyond the computational power of the decision-maker, or the sequence of implications stretches so far into the future that the consequences get shrouded in the mists of time" (Hay, 1983, p. 137).

Friedmanite defence of optimization, which heavily depends on "natural selection", "intuition" and "practice and/or learning" cannot save the optimal control applications from the failure in dynamic optimisation of consumption. Recall that in an optimal control of rational expectations permanent income/life cycle hypotheses, the optimal current consumption is contingent on future optimal decisions

on consumption. Neither practice nor learning from past mistakes can make it easier to compute the optimal consumption sequence. Although the Euler equation, which is the necessary condition of optimisation, gives an intuitive meaning on balancing the marginal utilities of consumption in two periods, it does not provide any corresponding intuition for the actual consumption decisions. Moreover, "every individual consumer has to solve his or her own, unique lifetime backward induction problems and no *as if* simplifications are available" (Pemberton, 1993, p. 5). Thus, Friedman's natural selection argument and the associated concept of *innate abilities,* do not apply to the dynamic optimisation of consumer behaviour.

7. Summary and Concluding Results

A brief analysis of the theoretical problems associated with liquidity constraints in households' consumption behaviour presented in section 2. By examination of the properties of the optimal consumption path when liquidity constraints are binding, I have produced a number of results, with and without time-varying interest rates. These results, which are presented in sections 3, 4 and 5 include the followings:

i) By using the method of dynamic programming together with the envelope theorem, I obtained the optimality condition in terms of the Lagrange multiplier associated with the liquidity constraints. This multiplier represents the amount by which the consumer's utility will change if current constraints on borrowing become relaxed by one unit.

ii) By using the generalised Hamiltonian function in the maximum principle, I have shown how liquidity constraints can directly affect consumption behaviour along the optimal path.

iii) Again, by using the generalised Hamiltonian, I have shown how the existence of liquidity constraints rejects the Hall's random walk hypothesis. We know that the effects of liquidity constraints on consumption are usually tested indirectly. Our result here is of prime theoretical value because, for the first time, it gives an explicit relationship between liquidity constraints and the random walk hypothesis.

iv) The above result has been obtained under the conventional assumption that an individual's net indebtedness is a constant function of his income. By using the generalised Hamiltonian, I have shown

how different formulations of liquidity constraints can, in principle, be handled in dynamic optimization problems of consumer choice.

v) I have shown, using the Hamiltonian approach, how time-varying interest rates can affect consumption variations along the optimal consumption trajectory.

vi) Using the Kuhn-Tucker conditions, I have obtained an explicit relation which demonstrates how the existence of liquidity constraints can reject the Hall's random walk hypothesis. The same result has been obtained earlier in this paper by using the generalised Hamiltonian function. The application of the Kuhn-Tucker conditions, however, provides a better insight into the possible interactions between time-varying interest rates and the utility discount rate. However, the generalised Hamiltonian function has much wider capabilities in treating different models of liquidity constraints.

vii) I have obtained an approximate negative relation between the coefficient of absolute risk aversion and the intertemporal elasticity of substitution. I used this relation to generalise the above results further. The results obtained are useful since they specify the followings:

1. How the time-varying interest rates affect optimal consumption through intertemporal elasticity of substitution which acts as a coefficient.

2. How the intertemporal elasticity of substitution, which has appeared as the coefficient of liquidity constraint, affects the optimal consumption behaviour. It should be noted that equation (46) has an interesting property: it simultaneously captures the effects of the following variables on the optimal consumption path: *i*) the pure preference parameters; *ii*) the interest rates variations; and *iii*) the structural parameters prevailing in the credit markets which are manifested in modelling of liquidity constraints.

Our analysis of the optimal consumption behaviour in a stochastic environment (section 6) shows that the departure from the certainty equivalence, and assuming that an individual consumer is facing an uncertain future income, poses serious problems in the application of optimal control theory to dynamic optimisation of consumption. Numerical complexities in calculating the optimal consumption paths with future uncertainties are beyond the capacity of a representative consumer. Hence, the backward induction procedure inherent in stochastic dynamic programming, as well as the iterative techniques associated with the maximum principle, may be unacceptable

optimisation methods because the relevant agent in the real world cannot identify and attain the solution.

Appendix 1
In this Appendix we derive the familiar Euler equation for optimal consumption by using the functional recurrence equation of dynamic programming together with the envelope relation.

We start by defining the time-varying optimal value function $V_t(A_t)$ as

$$(47) \qquad V_t(A_t) = \max E\left[\sum_{r=t}^{T} (1+\delta)^{-(\tau-t)} U(C_\tau) \mid \Omega_t \right],$$

subject to the equation of motion, i.e.

$$A_{t+1} = (1+r)A_t + Y_t - C_t.$$

All the terms are defined earlier. Equation (47) implies that the optimal value function is the present discounted value of expected utility evaluated along the optimal trajectory. For example,

$$V_t(A_t) = \max\left\{ E[U(C_t) + (1+\delta)^{-1} U(C_{t+1}) + \cdots + (1+\delta)^{-(T-t)} U(C_T)] \mid \Omega_t \right\},$$

or

$$V_{t+1}(A_{t+1}) = \max\left\{ E[(1+\delta)^{-1} U(C_{t+1}) + (1+\delta)^{-2} U(C_{t+1}) + \cdots + (1+\delta)^{-(T-t)} U(C_T)] \mid \Omega_t \right\}.$$

Note that A_t is the only state variable in the model which is being directly affected by the only control variable in the model, i.e. the consumption C_t. Thus, the optimal value function is a function of the asset variable only. One can argue that the optimal value function is a function of conditional joint distribution of future labour income and rates of return. However, because our equation of motion for A_{t+1} does not allow for the impact of consumption on conditional joint distribution of future income (or the rate of return), these variables cannot be included as arguments in the optimal value function. The

fact that the optimal value function is time variant captures the possibility of having different forms of this function over time.

Using equation (47), we write the functional recurrence equation as follows,

$$(48) \qquad V_t(A_t) = \max_{C_t} \left\{ U(C_t) + (1+\delta)^{-1} E[V_{t+1}(A_{t+1})] \mid \Omega_t \right\}.$$

Equation (48) is based on Bellman's principle of optimality. To maximise the right hand side of equation (48) subject to our equation of motion, we take the derivative of the right hand side with respect to C_t. This gives

$$\frac{\partial U(C_t)}{\partial C_t} + (1+\delta)^{-1} E \frac{\partial V_{t+1}(A_{t+1})}{\partial A_{t+1}} \cdot \frac{\partial A_{t+1}}{\partial C_t} = 0.$$

Using the equation of motion to evaluate $\dfrac{\partial A_{t+1}}{\partial C_t}$, we find

$$U'(C_t) - (1+\delta)^{-1} E V'_{t+1}(A_{t+1}) = 0,$$

or

$$(49) \qquad\qquad U'(C_t) = (1+\delta)^{-1} E V'_{t+1}(A_{t+1}).$$

The functional form of the value function in equation (49), which is the first order condition for optimality, is not known. We, therefore, cannot make any significant progress by using equation (49). However, we can use the envelope relation between $U'(C_t)$ and $V'(A_t)$ along the optimal trajectory. Consider a small variation in A_t in equation (48). We have

$$V'(A_t) = (1+\delta)^{-1} E \frac{\partial}{\partial A_t} [V_{t+1}(A_{t+1})],$$

or

$$V'(A_t) = (1+\delta)^{-1} E \frac{\partial V_{t+1}(A_{t+1})}{\partial A_{t+1}} \cdot \frac{\partial A_{t+1}}{\partial A_t},$$

which by using the equation of motion becomes

$$V'(A_t) = (1+\delta)^{-1} EV'_{t+1}(A_{t+1})(1+r).$$

Upon using equation (49) we have

(50)
$$V'(A_t) = (1+r)U'(C_t).$$

According to the envelope relation (50), the marginal value of financial assets along the optimal trajectory, is equal to the marginal utility of consumption multiplied by $(1+r)$. In other words, the marginal value of financial assets is equal to the increase in marginal utility of consumption at time $t+1$ viewed as of time t.

The appearance of $(1+r)$ in envelope relation (50) refers to the fact that the term C_t has entered the equation of motion at the end of period t while the financial asset A_t is assumed to be known at the beginning of the period. Hence, A_{t+1} in the equation of motion, is a function of $(1+r)A_t$, whereas $Y_t - C_t$ is not multiplied by $(1+r)$. If we model the equation of motion as

(51)
$$A_{t+1} = (1+r)(A_t + Y_t - C_t),$$

the envelope relation would then appear as

(52)
$$V'(A_t) = U'(C_t).$$

Substituting the envelope relation (50) into the first-order conditions, equation (49), yields

$$U'(C_t) = (1+\delta)^{-1} EU'(C_{t+1})(1+r),$$

or

(53)
$$U'(C_t) = \frac{1+r}{1+\delta} EU'(C_{t+1}).$$

Equation (53) is the *Euler Equation* for the optimal consumption-saving problem. Note that changing the equation of motion to equation (51) and using the envelope relation (52), will not alter the Euler equation.

The Euler equation (53) is nothing but the generalization of Keynes-Ramsey condition under uncertainty; that is, the marginal rate of substitution between consumption in two periods is equal to the marginal rate of transformation. According to the Euler equation, if a consumer at time t reduces the consumption by ΔC and invests the resulting saving at the rate of interest i and consumes the proceeds at time $t+1$, then the decrease in utility at time t, which is $U'(C_t)$, must be equal to the increase in the expected utility at time $t+1$, i.e. $(1+r)EU'(C_{t+1})$, viewed as of time t, i.e. $(1+\delta)^{-1}\big[(1+r)EU'(C_{t+1})\big]$.

It is interesting to note that the Hall's random walk hypothesis, which is based on life-cycle/permanent-income hypothesis, can be directly derived from the Euler equation which is based on the dynamic programming, [Blanchard and Fisher (1990)]. To see this, it suffices to write equation (53) as follows.

$$\frac{1+r}{1+\delta}U'(C_{t+1}) = U'(C_t) + v_{t+1},$$

where it is assumed that $E(v_{t+1}) = 0$. Alternatively,

(54) $$U'(C_{t+1}) = \gamma U'(C_t) + \xi_{t+1},$$

where, $\gamma = \dfrac{1+\delta}{1+r}$. Moreover, under certain conditions, we can deduce from equation (54) that consumption follows a martingale process. For example, assuming a quadratic utility function together with the assumption that the rate of interest r is equal to the rate of time preference δ, will result the followings: *i*) $U'(C_t)$ and $U'(C_{t+1})$ become linear functions in consumption and *ii*) $\dfrac{1+\delta}{1+r}$ becomes 1 in equation (54). Thus, we can write

(55) $$C_{t+1} = C_t + \xi_{t+1}^*,$$

where $\xi_{t+1}^{*} = k\xi_{t+1}$ and k is a constant.

Appendix 2
For applying the method of maximum principle in its continuous version, we first write the *equation of motion* or *system dynamics,* as follows,

$$A_{t+1} - A_{t} = rA_{t} + Y_{t} - C_{t}.$$

The limit of the above equation when we take discrete periods of length Δt and then let Δt tend to zero, provides a continuous version of the equation of motion. Note that since our flow variables, rA_{t}, Y_{t} and C_{t}, are now rates per unit of time, the right hand side of the above equation should be multiplied by Δt. We, therefore, have

$$A(t + \Delta t) - A(t) = \left[rA(t) + Y(t) - C(t)\right]\Delta t.$$

Dividing by Δt and letting Δt go to zero, will give the time derivative of $A(t)$, i.e.

$$\dot{A}(t) = rA(t) + Y(t) - C(t).$$

Conventionally, we write t as a subscript in discrete models and as an argument in continuous cases. It is further assumed that the initial level of consumer's financial asset is given, i.e.

$$A(t_{0}) = A(0) = \alpha.$$

References

Bencivenga, Valerie R. and Bruce D. Smith (1991), "Financial intermediation and endogenous growth", *Review of Economic Studies,* vol. *LVII,* pp. 195-209.

Bertsekas, Dimitri (1976), *Dynamic Programming and Stochastic Control.* New York: Academic Press.

Besley, Timothy (1993), "Saving, credit and insurance", Research Programme in Development Studies, Centre for International Studies, Woodrow Wilson School, Princeton University. Discussion paper no. 167, February. Also in volume 3 of the

Handbook of Development Economics, edited by J. Behrman and T. N. Srinivasam.

Beyoumi, Tamim (1991), "Financial deregulations and household behaviour", Bank of England, mimeo, November.

Blanchard, Oliver Jean and Stanley Fischer (1990), "Optimal consumption, investment and inventory behaviour", in their book: *Lectures in Macroeconomics,* The MIT Press, pp. 275-319.

Campbell, John Y. and N. G. Mankiw (1989), "Consumption, income and interest rates: reinterpreting the time series evidence", NBER Working Paper, no. 2924. Also published with discussants comments in Oliver J. Blanchard and Stanley Fischer, *NBER Macroeconomics Annual,* Cambridge, Massachusetts: MIT Press, pp. 185-216.

Campbell, John Y. and N. G. Mankiw (1991), "The response of consumption to income: a cross-country investigation", *European Economic Review,* vol. 35, pp. 715-721.

Carroll, Christopher D. (1992), "The buffer-stock theory of saving: some macro-economic evidence", *Brookings Papers on Economic Activity,* pp. 61- 156.

Carroll, Christopher D. (1994), "How does future income affect current consumption", *Quarterly Journal of Economics,* vol. 109, February, pp. 111-147.

Cushing, M. J. (1992), "Liquidity constraints and aggregate consumption behaviour", *Economic Inquiry,* vol. 30, no. 1, pp. 134-153.

Deaton, Angus (1991), "Saving and liquidity constraint", *Econometrica,* vol. 59, pp. 1221-1248.

Dreze, Jacques and Franco Modigliani (1972), "Consumption decisions under uncertainty", *Journal of Economic Theory,* vol. 5, pp. 308-335.

Fischer, Stanley (1980), *Rational Expectations and Economic Policy.* University of Chicago Press.

Flavin, Marjorie (1993), "The excess smoothness of consumption: identification and interpretation", *Review of Economic Studies,* vol. 60, pp. 651-666.

Flavin, Marjorie A. (1985), "Excess sensitivity of consumption to current income: liquidity constraints or myopia ?", *Canadian Journal of Economics,* vol. 18, pp. 117-136.

Flavin, Marjorie, A. (1982), "The adjustment of consumption to changing expectations about future income", *Journal of Political Economy,* vol. 89, no. 5, pp. 974-1009.

Fry, Maxwell J. (1984), "Saving, financial intermediation and economic growth in Asia", *Asian Development Review,* vol. 2 no. 1, pp. 82-91.

Greenwood, Jeremy and Boyan Jovanovic (1990), "Financial development, growth and the distribution of income", *Journal of Political Economy,* vol. CXVIII, pp. 1076-1107.

Hakansson, Nils H. (1970), "Optimal investment and consumption strategies under risk for a class of utility functions", *Econometrica,* vol. 38, pp. 587607.

Hall, Robert E. (1978), "Stochastic implications of the life-cycle-permanent income hypothesis: theory and evidence", *Journal of Political Economy,* vol. 86, no. 6, pp. 971-978.

Hall, Robert E. (1985), "Real interest and consumption", Working Paper, NBER, no. 1694, August.

Hall, Robert E. (1988), "Intertemporal substitution in consumption", *Journal of Political Economy,* vol. 96, pp. 339-357.

Hall, Robert E. and Frederic S. Mishkin (1982), "The sensitivity of consumption to transitory income: estimates from panel data on households", *Econometrica,* vol. 50, no. 2, March, pp. 461-481.

Hay, J. D. (1983), "Whither uncertainty?" *Economic Journal,* vol. 93 (conference supplement), pp. 129-138.

Hayashi, Fumio (1985), "The effect of liquidity constraints on consumption: a cross-sectional analysis", *Quarterly Journal of Economics,* vol. 100, pp. 183-206.

Hayashi, Fumio (1987), "Tests for liquidity constraints: a critical survey and some new observations", Chapter 13 in *Advances in Econometrics,* edited by Truman F. Bewley, pp. 91-120.

Hayashi, Fumio, Takatoshi Ito and Joel Slemrod (1988), "Housing finance imperfections, taxation and private saving: a comparative simulation analysis of the United States and Japan", *Journal of the Japanese and International Economies,* vol 2, pp. 215-238.

Heller, Walter Perrin and Ross M. Starr (1979), "Capital market imperfection, the consumption function and the effectiveness of fiscal policy", *Quarterly Journal of Economics,* August.

Hubbard, R. Glenn and Kenneth L. Judd (1986), "Liquidity constraints, fiscal policy and consumption", *Brookings Papers on Economic Activity,* vol. 1, pp. 1-5l.

Jappelli, Tulio (1990), "Who is credit constrained in the US economy", *Quarterly Journal of Economics,* pp. 219-234.

Jappelli, Tulio and Marco Pagano (1994), "Saving, growth and liquidity constraints", *Quarterly Journal of Economics,* pp. 83-109.

Kimbell, Miles, (1988), "Precautionary saving and the marginal propensity to consume", manuscript, University of Michigan.

King, M. (1986), "Capital market imperfections and the consumption function", *Scandinavian Journal of Economics,* vol. 88, pp. 59-80.

Leland, H. E. (1968), "Saving and uncertainty: the precautionary demand for saving", *Quarterly Journal of Economics,* vol. 82, pp. 465-473.

Lucas, Robert E. (1976), "Econometric policy evaluation: a critique", in Karl Brunner and Allen H. Meltzer (eds.), *The Phillips Curve and Labour Markets,* Carnegie-Rochester Conference Series on Public Policies. vol. 1. Amsterdam: North-Holland Publishing *Co.,* pp. 19-46.

Mankiw, G. N., J. Rotemberg and 1. Summers (1985), "Intertemporal substitution in macro-economics", *Quarterly Journal of Economics,* vol. 100, pp. 225-251.

Mankiw, N. G. (1989), "Real business cycles: a new Keynesian perspective", *Journal of Economic Perspectives,* vol. 3, pp. 79-90.

Mariger, Randall P. (1986), *Consumption Behaviour and the Effects of Government Fiscal Policies.* Cambridge, Massachusetts: Harvard University Press, 265 pages.

McKinnon, Ronald 1. (1973), *Money and Capital in Economic Development,* Washington, D.C.: Brookings Institution.

Merton, Robert C. (1971), "Optimal consumption and portfolio rules in a continuous time model", *Journal of Economic Theory,* vol. 3, pp. 373-413.

Miller, Bruce 1. (1976), "The effect on optimal consumption of increased uncertainty in labour income in the multi-period case", *Journal of Economic Theory,* vol. 13, pp. 154-167.

Muellbauer John and Ralph Lattimore (1995), "The consumption function: a theoretical and empirical overview", in *Handbook of Applied Econometrics,* edited by M. Hashem Pesaran and R. Wickens, Oxford: Blackwell, pp. 221-331.

Muellbauer, John (1983), "Surprises in the consumption function", *Economic Journal,* Conference papers, pp. 34-49.

Muellbauer, John and Anthony Murphy (1990), "The UK current account deficit", *Economic Policy,* October, pp. 347-396.

Patterson, K. D. and B. Pesaran (1992), "The intertemporal elasticity of substitution in consumption in the US and the UK", Discussion Paper in Quantitative Economics and Computing, Department of Economics, University of Reading, Ser. E. vol. 1, no. 1, April.

Pemberton, James (1993), "Attainable non-optimality or unattainable optimality: a new approach to stochastic life cycle problem", *Economic Journal,* vol. 103, January, pp. 1-20.

Pontryagin, 1. S., V. G. Boltyanskii, R. V. Gamkrelidze and E. F. Miskchenko *(1955-1959) The Mathematical Theory of Optimal Processes,* Translated by K. N. Trirogoff. New York: Wiley, 1962. Pontryagin published a number of papers on optimal processes with his collaborators during 19551959, and published the complete proof of his maximum principle in a paper entitled "Optimal control processes" published in *Uspekhii Matematicheskikh Nauk* vol. 14, no. 1, 1959, pp. 3-20; the English translation of this paper has appeared in *Pontryagin's Selected Works,* published by Gordon and Breach, Science Publisher, London, volume 1, 1986, pp. 511-543.

Roell, A. (1986), "Capital market imperfections and the excess sensitivity of consumption to transitory income", Manuscript, London School of Economics.

Rothsdild, Michael and Joseph E. Stiglitz (1971), "Increasing risk II: its economic consequences", *Journal of Economic Theory,* vol. 3, pp. 66-84.

Sandmo, A. (1970), "The effects of uncertainty on saving decisions", *Review of Economic Studies,* vol. 37, pp. 353-360.

Selden, Larry (1978), "A new representation of preference over 'certain X uncertain' consumption pairs: the 'ordinal certainty equivalence' hypothesis", *Econometrica,* vol. 46, September, pp. 1045-1060.

Sibley, David S. (1975), "Permanent and transitory income effects in a model of optimal consumption with wage income uncertainty", *Journal of Economic Theory,* vol. 11, pp. 68-82.

Zeldes, Stephen (1989), "Consumption and liquidity constraints: an empirical investigation", *Journal of Political Economy,* vol. 46, pp. 305-346.

Zeldes, Stephen P. (1989), "Optimal consumption with stochastic income: deviations from certainty equivalence", *Quarterly Journal of Economics,* vol. 104, May, pp. 275-298.